ESSAYS ON SOCIALLY RESPONSIBLE OPERATIONS

A Dissertation
Presented to
The Academic Faculty

By

Can Zhang

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Industrial and Systems Engineering

Georgia Institute of Technology

August 2018

ESSAYS ON SOCIALLY RESPONSIBLE OPERATIONS

Approved by:

Dr. Turgay Ayer, Co-advisor
H. Milton Stewart School of Industrial and Systems Engineering
Georgia Institute of Technology

Dr. Chelsea C. White III,
Co-advisor
H. Milton Stewart School of Industrial and Systems Engineering
Georgia Institute of Technology

Dr. Atalay Atasu
Scheller College of Business
Georgia Institute of Technology

Dr. Martin Savelsbergh
H. Milton Stewart School of Industrial and Systems Engineering
Georgia Institute of Technology

Dr. L. Beril Toktay
Scheller College of Business
Georgia Institute of Technology

Date Approved: April 20, 2018

To my grandparents and my aunt.

ACKNOWLEDGEMENTS

First, I would like to express my sincere gratitude to my advisors Dr. Turgay Ayer and Dr. Chelsea C. White III. I am very fortunate to be advised by them during my Ph.D. years. Dr. Ayer is a great role model in many aspects. I tremendously benefited from his vision as well as the exposure to a diverse set of research topics I had while working with him. I am also very grateful for his thoughtful advice from detailed suggestions for paper-writing to high-level career guidance. Dr. White is a nicest professor I have ever met. His patience, encouragement and support have been key for me to conquer all the difficulties. His continued productivity, success in all aspects of his career, as well as his passion in cultivating the young generation also motivate me to move towards my goals.

My deepest gratitude also goes to my committee members and co-authors Dr. Atalay Atasu and Dr. L. Beril Toktay. They have provided me with new perspectives on the operations management area, and have been critical for me to identify my genuine interest in problems that have a social perspective. It was truly a privilege to be able to work with them and have their support during my Ph.D. studies. I also very much appreciated their trust and the opportunities they provided me with to teach in both undergraduate and MBA classes at Scheller. These opportunities equipped me well with both experience and confidence in teaching, and have been particularly valuable in my job search.

I would also like to sincerely thank Dr. Martin Savelsbergh for making the time to serve on my thesis committee and for providing detailed and highly constructive comments. His suggestions have been instrumental for me to improve my thesis.

I am also fortunate to have the support from other faculty members, including but not limited to Dr. Alejandro Toriello and Dr. Benoit Montreuil from the H. Milton Stewart School of Industrial and Systems Engineering, and Dr. Karthik Ramachandran, Dr. Morvarid Rahmani and Dr. Necati Tereyagoglu from the Scheller College of Business. I want to sincerely thank all of them for their support and advice.

Industry collaboration has been a critical part of my thesis work and all other work I conducted during my Ph.D. years. I would like to express my sincere thanks to Mr. Charles Redding (CEO of MedShare), Dr. John D. Roback (Medical Director of Emory University Hospital Blood Bank). Mr. Kevin Lee (Process Optimization Senior Director of American Red Cross) and Ms. Mary Deck (Chief Manufacturing Executive of American Red Cross) for their trust and the wonderful opportunities to collaborate.

The support from all of my friends is also indispensable. I want to sincerely thank my friends and fellow students at Georgia Tech, including but not limited to Xin Wei, Yuan Li, Zhihao Ding, Senghan Lee, Ning Tian, Yanling Chang, Chenxi Zeng, Satya Malladi, Qiushi Chen, Jan Vlachy, Anthony Bonifonte, Caglar Caglayan, Zhaowei She, Rui Gao, Weijun Xie, Xiaowei Yue, Xiaolei Fang, Yu Cao and Wenjia Wang. My life here would be a lot harder without them. I also appreciate the constant support from my college friends, Yiwen Zhu, Zhe Han, Huajun Chai, Kelong Xu, Xiaoxu Ning, Zhenchao Yuan, and middle/high school friends, Zhu Zhang, Lingao Su, Yongfeng Xu and Caiyun Zhu.

I would also like to thank Dr. Quanwang Li, my undergraduate research advisor at Tsinghua University, for his continued support. He always gives me the warmest welcome each time I goes back to Tsinghua. My sincere thanks also goes to Dr. Shiming Zhang, who I met in Liuyang in 2008. He treats me as a family member ever since.

Last and definitely not the least, my deep thanks goes to all of my family members. I grew up with my grandparents, Wanxin Feng and Ailian Zhang. It is their love, vision and determination that carried me through all the difficulties and made me who I am today. I cannot thank them enough. I also want to sincerely thank my aunt Jinling Zhang and uncles Jincheng Feng and Yicheng Feng, who always treat me like their own son. I am also very glad that I have three wonderful cousins, Huizhen Wang, Yanbo Feng and Yushun Feng, who are truly like my sister and brothers. My special thanks goes to my mom who always loved my unconditionally – too unfortunate she cannot see me graduate.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

SUMMARY

This dissertation consists of three essays on socially responsible operations. The unifying theme is the focus on nonprofits and healthcare supply chains. In particular, the first two essays (Chapters 2 and 3) study perishable inventory management problems motivated by blood supply chain management in a local hospital network, while the third essay (Chapter 4) studies a resource allocation problem faced by nonprofit humanitarian organizations that collect and deliver unused or reusable medical surplus products to underserved healthcare facilities in developing countries. The overarching objectives are to 1) develop (near) optimal and implementable policies that can help improve the effectiveness and efficiency of daily operations in health and humanitarian organizations; and 2) derive insights to shed light into key trade-offs in complex managerial decisions in those settings.

CHAPTER 1
INTRODUCTION

In the past few years, there has been an increasing interest in studying operations management problems that have a societal perspective or aim to address social issues. Typical examples include many activities undertaken by nonprofit organizations with the objective to better serve the underserved. Problems in these contexts face unique challenges that significantly differ from those in traditional for-profit settings. For example, the objective of these problems is to improve social welfare or service levels to beneficiaries instead of maximizing profit, and monetary transfer is usually not used for matching supply with demand (e.g., beneficiaries may not pay for the products or services). Also, nonprofit organizations typically face tight operational and cash constraints, which requires innovative solution methods to allocate the scarce resource in an effective and efficient manner. Finally, implementation in these contexts is challenging due to the possible lack of operational expertise therein, which hence requires not only competitive performance but also simplicity of the proposed solution approaches to facilitate real-world implementation.

This dissertation studies three important problems faced by nonprofits and healthcare supply chains, and our contributions are mainly the following two folds: First, we develop provably good and easy-to-implement policies to support daily decision making in the health and humanitarian contexts, and work closely with our industry collaborators to implement our proposed solutions. For example, we have been working with the Emory University Hospital to implement our proposed inventory ordering and sharing policies for their management of platelet inventory, and working with MedShare, a top-ranked Southern-US based medical surplus recovery organization, to implement our proposed scoring mechanism to support their recipient selection and biomedical equipment allocation decisions. Second, we derive insights to shed light into the key trade-offs in complex managerial decisions in these settings. For example, by studying inventory sharing in a blood supply chain, we identify managerial insights that are significantly different from common wisdom derived from studies on traditional nonperishable inventory sharing.

In particular, motivated by a platelet inventory management problem in a local hospital, Chapter 2 of the dissertation considers a periodic-review, fixed-lifetime perishable inventory management problem where demand is a general stochastic process. Determining an optimal solution for this problem is intractable due to the ”curse of dimensionality”. In this paper, we first present a computationally efficient algorithm that we call the marginal-cost dual-balancing policy for the perishable inventory management problem. We then prove that a myopic policy under the so-called marginal-cost accounting scheme provides a lower bound on the optimal ordering quantity. By combining the specific lower bound we derive and any upper bound on the optimal ordering quantity with the marginal-cost dual-balancing policy, we present a new algorithm that we call the truncated-balancing policy. We prove that when first-in-first-out (FIFO) is an optimal issuing policy, the expected total cost of our policies is at most twice that of an optimal ordering policy. Finally, we conduct numerical analyses based on real data and show that both of our algorithms perform much better than the worst-case performance bound, and the truncated-balancing policy has a significant performance improvement over the balancing policy.

Chapter 3 considers inventory sharing in a two-location perishable inventory system, specifically in the context of platelet inventory sharing in a two-location hospital network. While the existing studies on inventory sharing have primarily focused on nonperishable products, motivated by the platelet inventory management problem, we present one of the first analyses of perishable inventory sharing in a two-location system. We assume that each location faces stochastic demand and replenishes its inventory using a base-stock policy. Under given base-stock levels, we determine the direction of transshipment and derive bounds on the optimal transshipment quantities, which enable us to develop an intuitive transshipment policy and derive approximations of the expected cost functions. Using extensive numerical analyses, we show that our proposed policy performs near-optimally. Comparing our results with those from the multi-location nonperishable and single-location perishable inventory literature, we further establish the following findings: i) Transshipments should occur more often (or with a larger quantity) when products are perishable; ii) unlike the well-established result in the single-location setting, in a two-location setting where inventory sharing is possible, it may be optimal to order strictly more in the perishable case than in the nonperishable case; and iii) contrary to the established finding in the nonperishable case, the value of inventory sharing for the perishable case can be substantial even when demand at one location is deterministic.

Chapter 4 considers a resource allocation problem faced by Medical Surplus Recovery Organizations (MSROs) that recover medical surplus products to fulfill the needs of under-served healthcare facilities in developing countries. Due to the uncertain, uncontrollable supply and limited information about recipient needs, delivering the right product to the right recipient in MSRO supply chains is particularly challenging. The objective is to identify strategies to improve the value provision capability of the MSROs. In particular, we propose a mechanism design approach to determine which recipient to serve at each shipping opportunity based on the recipients reported rankings of products. We find that when MSRO inventory information is shared with recipients, the only truthful mechanism is random selection among the recipients. We then show that eliminating the inventory information provision and withholding information regarding other recipients both enlarge the set of truthful mechanisms, thereby increasing the total value provision. Further, we show that under a wide class of implementable mechanisms, eliciting valuations has no value-added beyond eliciting rankings. Finally, we present a calibrated numerical study based on historical data from a partner MSRO, and show that a strategy consisting of a ranking-based mechanism in conjunction with eliminating inventory and competitor information can significantly improve the value provision for the MSROs.

CHAPTER 2

2-APPROXIMATION ALGORITHMS FOR PERISHABLE INVENTORY MANAGEMENT WHEN FIFO IS AN OPTIMAL ISSUING POLICY

2.1 Introduction.

Typical examples of perishable products include medical products such as blood and certain pharmaceuticals, and food products such as refrigerated meat and many dairy products. Unlike nonperishable products that can wait in inventory indefinitely until they are used to satisfy demand, perishable products must be used within a short period of time and will become outdated otherwise. Outdating can result in a significant amount of waste and financial loss. For example, the number of platelets outdated in 2011 in the U.S. was approximately 321,000 units, which accounted for 12.8% of all processed units ([1]). Similarly, the total annual unsaleable costs in the food, beverage, health and beauty industries in the U.S. were estimated as $15 billion, and about 17% of these costs (over 2.5 billion dollars) were caused by outdating ([2]). These facts underline the critical need for efficient inventory management policies for perishable products.

Our study is specifically motivated by a platelet inventory management problem faced by a local acute-care hospital, Emory University Hospital Midtown. In Emory Midtown, the demand for platelets mainly comes from cardiac surgeries, which account for more than 85% of its platelet transfusion. In this case, the uncertainty of demand stems from two sources: i) the number of surgeries performed per day, and ii) the amount of platelets needed per surgery. Such a structure of demand is common for many blood products. As such, the compound Poisson distribution, where a random (Poisson) amount of patients arrive at every time period (e.g., day) and each patient consumes a random amount of blood products, has been widely assumed for modeling demand in the blood supply chain literature (e.g., [3, 4, 5]). However, while simply assuming random arrivals of patients is reasonable for some cases such as trauma patients, more detailed forecast information on the number of arrivals is often available for many other cases, especially for scheduled operations such as cardiac surgeries. Most of those surgeries are scheduled days or even weeks in advance, and hence the number of surgeries that will be performed each day is gradually revealed as time approaches. Although the compound structure of demand is widely considered in the blood inventory literature, to our knowledge, the gradually revealed forecast information on the number of arrivals has not been formally captured.

Motivated by the platelet inventory management problem at Emory Midtown, in this paper, we study a periodic-review, fixed-lifetime perishable inventory problem under a general demand process, which can be nonstationary, correlated, and dynamically evolving over time. Similar to many other perishable inventory studies, we consider the first-in-first-out (FIFO) issuing policy, i.e., older products are issued first to meet demand, which is shown to perform very well in many perishable inventory systems (e.g., [6, 7]).

Our contributions in this paper are as follows:

i) We present an approximation algorithm that we call the *marginal-cost dual-balancing policy* (or the balancing policy for short) for the perishable inventory problem. We prove that when FIFO is an optimal issuing policy, our algorithm has a worst-case performance guarantee of two, i.e., the expected total cost of our policy is at most twice that of an optimal policy.

ii) While it is intuitive that FIFO helps reduce the amount of outdate by removing the oldest products from inventory, it is not obvious when it is guaranteed to be an optimal issuing policy. In this regard, we extend existing results in the literature on the optimality of FIFO and provide a necessary and sufficient condition and an easy-to-check sufficient condition for FIFO to be optimal.

iii) We present a tight example to show that the worst-case performance bound of two for the balancing policy can be achieved asymptotically when the unit shortage penalty goes to infinity, in which case the balancing policy tends to under-order. We also present a counterexample to show that when FIFO is not optimal, the worst-case performance bound can be strictly greater than 2.

iv) We derive a lower bound on the optimal ordering quantity by minimizing the one-period cost under the so-called marginal-cost accounting scheme. Then, by "truncating" the balancing quantity using this (or any looser) lower bound and any upper bound on the optimal ordering quantity, we present a new algorithm that we call the *truncated-balancing policy*. We prove that the truncated-balancing policy also admits a worst-case performance guarantee of two when FIO is optimal.

v) Lastly, we conduct extensive numerical analyses using both hypothetical and real data to show that a) our proposed algorithms perform significantly better than the worst-case performance bound of two; and b) the truncated-balancing policy performs significantly better than the balancing policy and existing policies in the literature, especially when the unit shortage penalty is large.

In the literature, many papers have studied the periodic-review, fixed-lifetime perishable inventory management problem (see reviews by [8, 9] and [10]). The general multi-period lifetime perishable inventory problem was first studied independently by [11] and [12], who both formulated the problem as a dynamic program (DP) with a state space comprised of inventory levels of different ages. However, the structure of an optimal policy is complicated and finding an optimal policy using standard dynamic programming is computationally intractable due to the well-known "curse of dimensionality". Therefore, later efforts are mainly focused on heuristic policies. Among the developed heuristic policies, the base-stock policy, under which the total inventory is replenished up to the same level (i.e., base-stock level) at each period, is particularly popular due to its simplicity and near-optimal numerical performance (e.g., [13, 14, 15, 16, 17, 18, 19, 20]). Other heuristic policies such as the modified base-stock policy (e.g., [21]), the constant order policy (e.g., [22, 23]), and a higher-order approximation ([24]) are also proposed and studied. However, due to the complexity of the perishable inventory management problem, most of these studies assume that demand over time is independently and identically distributed (i.i.d.), and none of the proposed heuristic policies has a theoretical performance guarantee.

More recently, there is a stream of work focusing on approximation algorithms for stochastic inventory systems under general demand processes. The pioneering work by [25] studies a stochastic inventory management problem for nonperishable products. They show that the proposed dual-balancing policy, which balances the costs of under-ordering and over-ordering under a marginal-cost accounting scheme, has a worst-case performance guarantee of two. This idea has been later extended to many other settings to consider lost sales ([26]), setup costs and capacity constraints ([27, 28, 29]), remanufacturing ([30]), and perishable products ([31, 32, 33]).

Among these papers that study approximation algorithms in inventory management, [32], which also considers a perishable inventory control problem with zero lead time and no set-up cost, is the most relevant to ours. In particular, [32] present a proportional-balancing policy and a dual-balancing policy for perishable inventory systems that follow FIFO issuing policy, and they prove that 1) the proportional-balancing policy has a performance guarantee between two and three for the general case, and 2) the dual-balancing policy has a performance guarantee of two when demand is independent and stochastically non-decreasing over time.

While both our study and [32] focus on developing approximation algorithms for perishable inventory systems, our analysis and results are different from theirs in the following aspects (see also Table 2.1): i) We present algorithms that are different from the ones presented in [32]. ii) We tighten the worst-case performance guarantee for perishable inventor systems to exactly two for cases where FIFO is an optimal issuing policy, and show that the condition presented in [32] to ensure a performance guarantee of two (i.e., demand is independent and stochastically non-decreasing) is a special case of ours. iii) We further consider truncating our balancing policy using a specific lower bound we derive on the optimal ordering quantity, and establish worst-case performance guarantee for the truncated-balancing policy. This is an important contribution because, unlike the existing results in the nonperishable inventory literature, truncation in the perishable inventory case imposes several new methodological challenges. Further, as we show in §4.7, the truncated-balancing policy performs much better than the existing policies in the literature. iv) Methodologically, while [32] build their analysis based on algebraic arguments, our analysis is based on two new ideas that we call the *imaginary operation policy* and the *dynamic unit-matching scheme*, respectively (see discussion also in the following paragraphs).

Table 2.1: A summary of the major differences of our work from [32]

Policies	Proportional-balancing ([32])	Dual-balancing ([32])	Marginal-cost dual-balancing (our work)
Common assumptions	FIFO issuing policy, no setup cost, zero lead time, backlogging/lost sales		
Specific settings	A: General	B: Demand is independent and stochastically non-decreasing	C: FIFO is an optimal issuing policy
Relationship	Setting B $\subset$ Setting C $\subset$ Setting A		
Performance guarantee	Between 2 and 3	2	2
Truncation of policies	No		Yes
Methodology	Algebraic arguments		Imaginary operation policy & Dynamic unit matching

Finally, we remark that the main challenge for the worst-case analysis for perishable inventory systems stems from the outdating process of perishable products. More specifically, the existing worst-case analysis for nonperishable inventory systems is based on a (static) one-to-one matching between units under the balancing policy and the optimal policy, which relies on the fact that all products will be used to satisfy demand (or remain in inventory at the end of the horizon). This is in contrast to the perishable inventory case where units can simply outdate without satisfying any demand. As [32] also point out, "the perishability of products destroys this matching mechanism, thus the existing techniques developed for non-perishable inventory systems are no longer applicable." In this paper, we introduce two new ideas: 1) an imaginary operation policy, under which old products can be replaced with new ones without cost so that the (otherwise partially ordered) inventory vectors under two different policies can be easily compared, and 2) a dynamic unit-matching scheme, under which units can be dynamically matched over time after we observe the realizations of outdates. These two ideas together allow us to address the challenges imposed by not only perishability but also truncation of balancing policies.

The remainder of this paper is organized as follows. In §2.2, we present a model formulation of the problem. In §2.3, we present a marginal-cost dual-balancing policy for the perishable inventory control problem. In §2.4, we prove that when FIFO is an optimal issuing policy, our algorithm has a worst-case performance guarantee of two, i.e., the expected total cost of our policy is at most twice that of an optimal ordering policy. We further compare our policy with base-stock policies and show that the expected total cost of our policy is always at most twice that of an optimal base-stock policy. In §2.5, we first show that a myopic policy under the marginal-cost accounting scheme provides a lower bound on the optimal ordering quantity; we then present a truncated-balancing policy that also admits a worst-case performance guarantee of two when FIFO is an optimal issuing policy. In §2.6, we present a necessary and sufficient condition and several easy-to-check sufficient conditions that ensure the optimality of FIFO issuing policy. Finally, we present computational results based on a platelet inventory control problem in §4.7, and draw conclusions in §4.8.

2.2 Model Formulation.

We study a periodic-review, fixed-lifetime perishable inventory management problem under a general stochastic demand process.

Notation. We consider a product lifetime of $K > 1$ periods and a planning horizon of T periods. Demands over the planning horizon are denoted as $D_1, ..., D_T$, which are exogenous random variables with finite means and can be nonstationary, correlated and dynamically evolving. As a convention, we generally use capital letters to denote random variables, and lowercase letters to denote their realizations (product lifetime K and planning horizon T are exceptions). At the beginning of each period t, there is an information set denoted as f_t, which contains the realization of demands $(d_1, ..., d_{t-1})$ and possibly some other forecast information available at period t, denoted as $(u_1, ..., u_t)$. That is, the information set f_t is a specific realization of the random vector $F_t = (D_1, ..., D_{t-1}, U_1, ..., U_t)$. Let $\mathscr{F}_t$ be the set of all possible f_t. Further, we assume that the conditional joint distribution of future demands $(D_t, ..., D_T)$ is known for given f_t. Additional notation that describes system states and decision variables is defined as follows:

$X_{k,t}$: the inventory level of age k at the beginning of period t, $k = 1, ..., K-1; t = 1, ..., T$.

$\mathbf{X}_t$: the inventory vector at the beginning of period t, i.e., $\mathbf{X}_t = (X_{1,t}, ..., X_{K-1,t})$, $t = 1, ..., T$.

Q_t: the ordering quantity at period $t, t = 1, ..., T$.

Y_t: the total inventory level after ordering and before demand realization at period t, i.e., $Y_t = \sum_{k=1}^{K-1} X_{k,t} + Q_t$, $t = 1, ..., T$.

System Dynamics. We define the sequence of events as follows: 1) At the beginning of each period $t = 1, ..., T$, the $K-1$ dimensional inventory vector $\mathbf{X}_t$ and the information set F_t are observed, based on which Q_t products of age zero are ordered; 2) products ordered arrive instantly with a zero lead time; 3) random demand D_t then occurs during the period, inventory is issued to satisfy demand based on the FIFO rule, and unmet demand is lost;[1] and 4) at the end of each period, all products in inventory age by 1, and products reaching age K are disposed from the inventory. Let $X_{0,t} = Q_t$. Then, the inventory vector is

[1]We note that since we assume zero lead time, our results hold equally well for the backlogging case. This is because with zero lead time, all backlogged demand can be satisfied at the beginning of the next period. Then, a backlogging model with cost parameters $\hat{c}, \hat{p}, \hat{h}$ and $\hat{w}$ is equivalent to a lost sales model with cost parameters $\hat{c}, \hat{p} + \beta\hat{c}, \hat{h}$ and $\hat{w}$.

updated as follows:

$$X_{k,t+1} = \left(X_{k-1,t} - \left(D_t - \sum_{m=k}^{K-1} X_{m,t}\right)^+\right)^+, k = 1, ..., K-1; t = 0, ..., T-1.$$

Without loss of generality, we assume that the system starts from empty (i.e., zero initial inventory); however, all of our results can be extended to consider arbitrary initial inventory levels.

Cost Structure. At each period, we consider an ordering cost $\hat{c}$ for each unit of product ordered at that period, a shortage penalty $\hat{p}$ for each unit of stock-out, a holding cost $\hat{h}$ for each unit of excess inventory after demand realization, and an outdating cost $\hat{w}$ for each unit of product that is outdated at the end of that period. To eliminate trivial situations, we assume $\hat{p} - \hat{c} \geq 0$. We allow negative outdating cost (i.e., salvage value) as long as $\hat{w} + \beta\hat{c} \geq 0$, where β denotes the discount factor. We also consider a salvage value for each unit of product left in inventory at the end of the planning horizon, and for simplicity we assume it is equal to the ordering cost $\hat{c}$ (our results can be easily extended to consider any salvage value $\hat{v}$ as long as $\hat{w} + \beta\hat{v} \geq 0$ and $\hat{h} + \hat{c} - \beta\hat{v} \geq 0$).

Optimality Criterion. At each period t, given the inventory vector $\mathbf{x}_t$ and the information set f_t, an ordering *decision rule* is a function that maps from the set of all possible $(\mathbf{x}_t, f_t)$ to the set of possible q_t; and an ordering *policy* is a collection of ordering decision rules at all periods. Let π denote any given ordering policy. Then, the total discounted cost under policy π is:

$$\hat{\mathscr{C}}(\pi) = \sum_{t=1}^{T} \beta^{t-1}\big(\hat{c}Q_t^\pi + \hat{p}(D_t - Y_t^\pi)^+ + \hat{h}(Y_t^\pi - D_t)^+ + \hat{w}(X_{K-1,t}^\pi - D_t)^+\big) - \beta^T \hat{c} \sum_{k=1}^{K-1} X_{k,T+1}^\pi,$$

where $\mathbf{X}_t^\pi$ and Q_t^π denote the inventory vector and ordering quantity at period t under policy π, respectively. Then, an optimal ordering policy OPT can be obtained by $OPT \in \arg\min_\pi \mathrm{E}[\hat{\mathscr{C}}(\pi)]$. We call the decision rules that constitute an optimal policy the optimal decision rules.

2.3 Marginal-Cost Dual-Balancing Policy.

In this section, we first introduce a cost transformation to eliminate ordering cost in §2.3.1. We then present a marginal-cost accounting scheme for the perishable inventory setting in §2.3.2. Finally, we present our algorithm in §2.3.3. Unless presented in the main text, the proofs of all analytical results are included in the appendix.

2.3.1 Cost Transformation.

To apply the marginal-cost accounting scheme which we present in §2.3.2, we first need to construct an equivalent problem with zero ordering cost.

Define the cost parameters for the transformed problem as: $c = 0, p = \hat{p} - \hat{c}, h = \hat{h} + (1-\beta)\hat{c}$, and $w = \hat{w} + \beta\hat{c}$. Since we assume $\hat{p} - \hat{c} \geq 0$ and $\hat{w} + \beta\hat{c} \geq 0$, all of the transformed costs are nonnegative. Then, for a given policy π, the total discounted cost of the transformed problem is:

$$\mathscr{C}(\pi) = \sum_{t=1}^{T} \beta^{t-1}\big(p(D_t - Y_t^\pi)^+ + h(Y_t^\pi - D_t)^+ + w(X_{K-1,t}^\pi - D_t)^+\big).$$

In the following lemma, we show that the difference between the total discounted costs of the original and transformed problems is independent of policy π, which implies that the two problems are equivalent in the sense that they have the same set of optimal ordering policies.

Lemma 1 *For any policy π, $\hat{\mathscr{C}}(\pi) - \mathscr{C}(\pi) = \sum_{t=1}^{T} \beta^{t-1}\hat{c}D_t$, with probability one.*

2.3.2 Marginal-Cost Accounting Scheme.

Unlike traditional methods which assign each period all costs that occur at this period, the marginal-cost accounting scheme, first introduced by [25], assigns each period all costs that are "caused" by the decision made at this period. For example, a unit ordered at period

t may stay in the system for multiple periods, thus holding costs may be charged for this unit for multiple periods; under the marginal-cost accounting scheme, all of these holding costs are assigned to period t. Following a similar logic, we next present the marginal-cost accounting scheme for the perishable inventory setting.

Marginal Shortage Penalty. Since inventory can be replenished with a zero lead time, any mistake of under-ordering at a given period can be fixed by ordering more in the next period. Hence, the marginal shortage penalty at each period is simply defined as the shortage penalty that occurs at this period. For $t = 1, ..., T$, given $\mathbf{x}_t$, f_t and q_t, let $P_t(\mathbf{x}_t, f_t, q_t)$ denote the expected marginal shortage penalty at period t. Then, we have:

$$P_t(\mathbf{x}_t, f_t, q_t) := \beta^{t-1} p \mathrm{E}[(D_t - y_t)^+ | f_t].$$

Marginal Holding Cost. For $t = 1, ..., T$, given $\mathbf{x}_t$, f_t and q_t, let $H_t(\mathbf{x}_t, f_t, q_t)$ denote the expected marginal holding cost at period t, which is defined as the sum of all expected holding costs charged for units ordered at period t. In the perishable inventory setting, since units in inventory may become outdated without satisfying any demand, the future holding costs charged for q_t depend on the entire inventory vector $\mathbf{x}_t$. Thus, similar to [12], we let $A_{0,t} = 0$, and for $k = 1, ..., K-1$, let $A_{k,t}$ be the total demand over periods $t, ..., t+k-1$ that cannot be satisfied by the inventory of $(x_{K-k,t}, ..., x_{K-1,t})$, i.e., the inventory that would have been outdated by the end of period $t+k-1$. Then:

$$A_{k,t} = (A_{k-1,t} + D_{t+k-1} - x_{K-k,t})^+, k = 1, ..., K-1.$$

Thus, for $k = 0, ..., K-1$, $(A_{k,t} + D_{t+k} - \sum_{m=1}^{K-k-1} x_{m,t})^+$ represents the total demand over periods $t, ..., t+k$ that cannot be satisfied by the inventory of $\mathbf{x}_t$, and $(q_t - (A_{k,t} + D_{t+k} - \sum_{m=1}^{K-k-1} x_{m,t})^+)^+$ represents the amount of q_t left in inventory at the end of period $t+k$.

Then, we have:

$$H_t(\mathbf{x}_t, f_t, q_t) := \sum_{k=0}^{K-1} \beta^{t+k-1} h\mathrm{E}\left[(q_t - (A_{k,t} + D_{t+k} - \sum_{m=1}^{K-k-1} x_{m,t})^+)^+ \middle| f_t \right],$$

where the sum over k is defined up to $T - t$ when $t + K - 1 \geq T$.

Marginal Outdating Cost. For $t = 1, ..., T$, given $\mathbf{x}_t$, f_t and q_t, let $W_t(\mathbf{x}_t, f_t, q_t)$ denote the expected marginal outdating cost at period t, which is defined as the sum of all expected outdating costs charged for units ordered at period t, i.e., the expected outdating costs that occur at period $t + K - 1$. Since units ordered at periods $T - K + 2, ..., T$ will not outdate within the planning horizon, we have $W_t(\mathbf{x}_t, f_t, q_t) = 0$ for $t \geq T - K + 2$. For $t \leq T - K + 1$, $(q_t - A_{K-1,t} - D_{t+K-1})^+$ represents the amount of q_t that will be outdated at the end of period $t + K - 1$. Then, we have:

$$W_t(\mathbf{x}_t, f_t, q_t) := \beta^{t+K-1} w\mathrm{E}[(q_t - A_{K-1,t} - D_{t+K-1})^+ | f_t].$$

For a given policy π, let P_t^π, H_t^π and W_t^π denote the corresponding marginal shortage penalty, holding and outdating costs at period t, respectively. Under a given policy π, $\mathbf{x}_t^\pi$ and q_t^π are both known for given f_t. Then, $\mathrm{E}[P_t^\pi | f_t] = P_t(\mathbf{x}_t^\pi, f_t, q_t^\pi)$, $\mathrm{E}[H_t^\pi | f_t] = H_t(\mathbf{x}_t^\pi, f_t, q_t^\pi)$, and $\mathrm{E}[W_t^\pi | f_t] = W_t(\mathbf{x}_t^\pi, f_t, q_t^\pi)$. Since the system starts from zero inventory, we have $\mathscr{C}(\pi) = \sum_{t=1}^{T} (P_t^\pi + H_t^\pi + W_t^\pi)$.

2.3.3 Algorithm.

Now we present our first algorithm based on the marginal-cost accounting scheme presented above. Clearly, the expected marginal shortage penalty $P_t(\mathbf{x}_t, f_t, q_t)$ occurs due to under-ordering, while the expected marginal holding and outdating costs $H_t(\mathbf{x}_t, f_t, q_t)$ and $W_t(\mathbf{x}_t, f_t, q_t)$ occur due to over-ordering. Therefore, we define the *marginal-cost dual-balancing policy* (denoted as B) as to balance the expected marginal shortage penalty against the sum of the expected marginal holding and outdating costs. More specifically, at each period t, given $\mathbf{x}_t$ and f_t, the marginal-cost dual-balancing ordering quantity q_t^B (for simplicity, we also call it the balancing quantity in the following text) is defined as the solution to the following equation:

$$P_t(\mathbf{x}_t, f_t, q_t^B) = H_t(\mathbf{x}_t, f_t, q_t^B) + W_t(\mathbf{x}_t, f_t, q_t^B). \tag{2.1}$$

Note that the existence of the balancing ordering quantity q_t^B is guaranteed, because at any period t, given $\mathbf{x}_t$ and f_t, $P_t(\mathbf{x}_t, f_t, q_t)$ is non-increasing in q_t; when $q_t = 0$, $P_t(\mathbf{x}_t, f_t, q_t)$ is nonnegative, and when q_t goes to infinity, $P_t(\mathbf{x}_t, f_t, q_t)$ goes to zero. In contrast, $H_t(\mathbf{x}_t, f_t, q_t)$ and $W_t(\mathbf{x}_t, f_t, q_t)$ are non-decreasing in q_t; when $q_t = 0$, $H_t(\mathbf{x}_t, f_t, q_t) = W_t(\mathbf{x}_t, f_t, q_t) = 0$, and when q_t goes to infinity, both $H_t(\mathbf{x}_t, f_t, q_t)$ and $W_t(\mathbf{x}_t, f_t, q_t)$ go to infinity. Therefore, when we allow fractional ordering quantities, q_t^B is guaranteed to exist. The algorithm can be easily extended to consider discrete ordering quantities following a similar argument as in [25].

We also remark that our marginal-cost dual-balancing policy is different from both the proportional-balancing policy and the dual-balancing policy defined in [32]. First, while our policy uses a balancing ratio of 1:1 (i.e., set the marginal shortage penalty to be equal to the sum of the marginal holding and outdating costs), the proportional-balancing policy in [32] uses a balancing ratio of $1 : \frac{Kh+w}{2(K-1)h+w}$. Second, while our policy balances the marginal shortage penalty against the sum of the marginal holding and outdating costs, the dual-balancing policy in [32] balances the marginal shortage penalty against the marginal outdating cost plus the holding cost that occurs at period t, i.e., the marginal holding cost $H_t(\mathbf{x}_t, f_t, q_t)$ in Equation (2.1) is replaced by $\beta^{t-1}h\mathrm{E}[(y_t - D_t)^+|f_t]$ in the dual-balancing policy in [32].

2.4 Worst-Case Analysis.

In this section, we first build a bridging policy in §2.4.1. Then, in §2.4.2, we construct a new unit-matching scheme that (dynamically) matches units under two different policies on a one-to-one correspondence. Based on these results, we prove the worst-case performance guarantee of our algorithm in §2.4.3 under the assumption that FIFO is an optimal issuing policy. Finally, in §2.4.4 and §2.4.5, we present a tight example and a counterexample, respectively, to show that our worst-case performance guarantee of two is tight, and that the worst-case performance bound can be strictly greater than two when FIFO is not optimal.

2.4.1 A Bridging Policy: Imaginary Operation Policy.

By Lemma 1, we know that $\hat{\mathscr{C}}(\pi) - \mathscr{C}(\pi)$ is nonnegative and independent of policy π. Therefore, to show that the expected total cost of the marginal-cost dual-balancing policy is at most twice that of an optimal ordering policy (i.e., $\mathrm{E}[\hat{\mathscr{C}}(B)] \leq 2\mathrm{E}[\hat{\mathscr{C}}(OPT)]$), it is sufficient to show that $\mathrm{E}[\mathscr{C}(B)] \leq 2\mathrm{E}[\mathscr{C}(OPT)]$.

However, due to the partially ordered nature of multi-dimensional inventory vectors in the perishable case, it is difficult to directly compare the costs under policies B and OPT. Therefore, we next propose a bridging policy that we call the *imaginary operation policy* (denoted as IM), which allows us to properly modify the inventory vectors so that the inventory vectors under two different policies become completely ordered, and the respective costs can be easily compared. We then show $\mathrm{E}[\mathscr{C}(IM)] \leq \mathrm{E}[\mathscr{C}(OPT)]$ (Lemma 4) and $\mathrm{E}[\mathscr{C}(B)] \leq 2\mathrm{E}[\mathscr{C}(IM)]$ (Lemma 6), respectively, which together lead to our main result $\mathrm{E}[\mathscr{C}(B)] \leq 2\mathrm{E}[\mathscr{C}(OPT)]$ (Theorem 1).

We start with defining the ordering rule under policy IM. At each period t, given the system state $\mathbf{x}_t$ and f_t, let the system under policy IM follow an optimal ordering decision rule (note that this is different from copying the ordering quantity from the system under policy OPT). What differentiates policies IM and OPT is that under policy IM, at each period after ordering and before demand realization, products in the inventory vector can be "moved" from older positions to the position of age zero, i.e., old products can be replaced with new ones for free.

We now define how units are moved under policy IM along a given sample path f_{T+1}. First, at period 1, let y_1^B and y_1^{IM} be the total inventory levels after ordering under policies B and IM, respectively. Suppose $y_1^B \geq y_1^{IM}$. Then no units are moved at this period. Suppose $y_1^B < y_1^{IM}$. Then we move all units in the inventory vector under policy IM to the position of age zero.

At period 2, let y_2^B and y_2^{IM} be the total inventory levels after ordering under policies B and IM, respectively. Suppose $y_2^B \geq y_2^{IM}$. Then no units are moved at this period. Suppose $y_2^B < y_2^{IM}$. Consider the following two subcases: i) if $y_1^B \geq y_1^{IM}$, i.e., no units were moved at period 1, then we move all units under policy IM to age zero; ii) if $y_1^B < y_1^{IM}$, i.e., all units under IM were moved to age zero at period 1, then the rule of movements for period 2 is defined as follows.

Since all units were moved to age zero at period 1, the inventory under policy IM is consumed (used to satisfy demand or outdated) no faster than that under policy B. Also, we have $y_1^{IM} > y_1^B$. Then at period 2, the total inventory of age greater than or equal to one under policy IM is no less than that under policy B, i.e., $\sum_{k=1}^{K-1} x_{k,2}^{IM} \geq \sum_{k=1}^{K-1} x_{k,2}^{B}$. Then, at the beginning of period 2 after ordering, we move some units of age one under IM to age zero such that $\sum_{k=1}^{K-1} x_{k,2}^{IM} = \sum_{k=1}^{K-1} x_{k,2}^{B}$.

Similarly, at any period t, let y_t^B and y_t^{IM} be the total inventory levels after ordering under policies B and IM, respectively (both are well defined after the rules of movements in periods $1, ..., t-1$ are defined). Then, we assign period t into one of the following subsets of {1,...,T}:

$$\mathscr{T}_P := \{t : y_t^B \geq y_t^{IM}\}, \mathscr{T}_H := \{t : y_t^B < y_t^{IM}\}.^2$$

[2]Note that to define the rule of movements in period t under policy IM, we do not need to know the full partition of the decision epochs $\{1, ..., T\}$; instead, we only need to know whether each period $1, ..., t$

Suppose $t \in \mathscr{T}_P$, i.e., $y_t^B \geq y_t^{IM}$. Then no units are moved at period t. Suppose $t \in \mathscr{T}_H$, i.e., $y_t^B < y_t^{IM}$. Let $\mathscr{T}_H = \{\tau_1, ..., \tau_n\}$, where $\tau_1 < ... < \tau_n$. Then, if $t = \tau_1$, we move all units under policy IM to age zero. If $t = \tau_i$ for some $i \geq 2$, we first move all units of age strictly less than $\tau_i - \tau_{i-1}$ under policy IM to age zero, and then for each $j = 1, ..., i-1$, move some units of age equal to $\tau_i - \tau_j$ (i.e., ordered in period τ_j) under policy IM to age zero such that after all the movements, we have:

(i) There is only positive inventory of age $0, \tau_i - \tau_{i-1}, ..., \tau_i - \tau_1$ under policy IM.

(ii) For $j = 1, ..., i-1$, the total inventory of age greater than or equal to age $\tau_i - \tau_j$ under policies IM and B are the same, i.e.,

$$\sum_{k=\tau_i-\tau_j}^{K-1} x_{k,\tau_i}^{IM} = \sum_{k=\tau_i-\tau_j}^{K-1} x_{k,\tau_i}^{B}, j = 1, ..., i-1. \tag{2.2}$$

By construction, we are ensured to have that: $\forall \tau_i \in \mathscr{T}_H$, after the movements of units at τ_i, the inventory vector under policy IM is "younger" than that under policy B, i.e., for $k = 1, ..., K-1$, policy IM has no more inventory of age greater than or equal to k.

Lemma 2 *$\forall \tau_i \in \mathscr{T}_H$, after the movements of units at τ_i, we have:*

$$\sum_{m=k}^{K-1} x_{m,\tau_i}^{IM} \leq \sum_{m=k}^{K-1} x_{m,\tau_i}^{B}, k = 1, ..., K-1. \tag{2.3}$$

An illustrative example describing the rules of movements is presented in Figure 2.1. In this example, we have product lifetime of $K = 3$ periods, and planning horizon of $T = 4$ periods. Consider a given sample path where $d_1 = d_2 = d_3 = 0$ and $d_4 = 2$. At the beginning of period $t = 1$, assume $q_1^B = 2$ and $q_1^{IM} = 1$. Then, $y_1^B = 2 > 1 = y_1^{IM}$, thus $t = 1 \in \mathscr{T}_P$ and no movements are performed at this period. At the beginning of period $t = 2$, assume $q_2^B = 1$ and $q_2^{IM} = 3$. Then, $y_2^B = 3 < 4 = y_2^{IM}$, thus $t = 2 = \tau_1 \in \mathscr{T}_H$, and we move all units under policy IM to age 0. At the beginning of period $t = 3$, assume $q_3^B = q_3^{IM} = 2$. Then, $y_3^B = 5 < 6 = y_3^{IM}$, thus $t = 3 = \tau_2 \in \mathscr{T}_H$. The unit ordered at τ_1

belongs to $\mathscr{T}_P$ or $\mathscr{T}_H$.

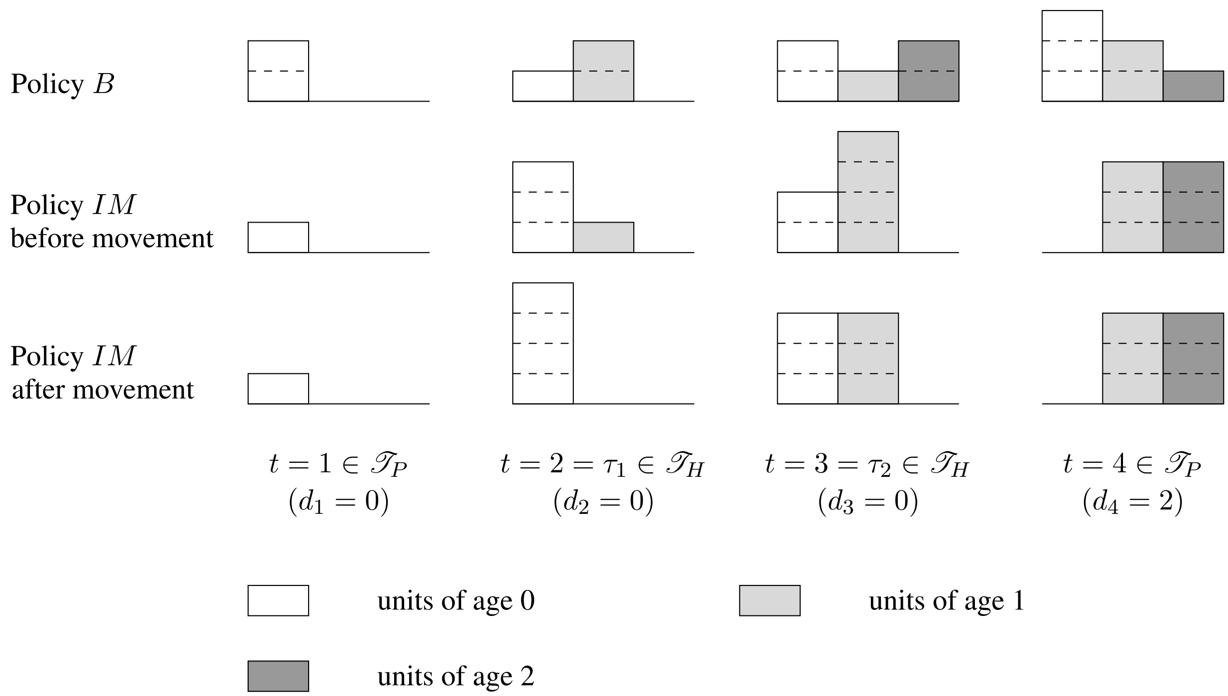


Figure 2.1: An illustrative example to show the imaginary operation policy (IM)

under policy B is now of age $\tau_2 - \tau_1 = 1$, therefore we move one unit of age 1 under policy IM to age 0 such that the amount of units of age greater than or equal to 1 under policies B and IM are equal. At the beginning of period $t = 4$, assume $q_4^B = 3$ and $q_4^{IM} = 0$. Then, $y_4^B = 6 = y_4^{IM}$, thus $t = 4 \in \mathscr{T}_P$ and no movements are performed at this period.

2.4.2 A Dynamic Unit-Matching Scheme.

Based on the imaginary operation policy (IM) we constructed above, we now introduce a new unit-matching scheme that matches inventory units under policies B and IM, which plays a key role in the comparison of $\mathscr{C}(B)$ and $\mathscr{C}(IM)$. In particular, our objective is to match the units ordered at each period $t \in \mathscr{T}_H$ under policy B to units under policy IM on a one-to-one correspondence, such that a matched unit under policy B stays in inventory no longer than the corresponding unit under policy IM. This way, the total holding and outdating costs charged for the units ordered at $t \in \mathscr{T}_H$ under policy B are bounded by the total holding and outdating costs under policy IM.

The idea of examining inventory and demand at a unit level is first proposed by [34], and is first applied to prove worst-case performance guarantee by [25], where units under two policies are matched on a one-to-one correspondence. Similar arguments are also used in all the subsequent studies on approximation algorithms for nonperishable inventory systems ([26, 27, 28, 29, 30]). However, in these studies, the matching of inventory units is static in the sense that once a pair of units under two policies are matched at some period, the matching is permanent. This approach relies on the assumption that all units ordered will be eventually used to satisfy demand (or remain in inventory at the end of the horizon), and a pair of units, once matched, will be used to satisfy the same unit of demand. However, this is in contrast to the perishable inventory setting where units in inventory may simply outdate without satisfying any demand. To address this complication, we introduce a new matching scheme that we call the *dynamic unit-matching scheme*, under which, a unit ordered at $t \in \mathscr{T}_H$ under policy B can be matched and then re-matched to a new unit under policy IM. The rules of matchings are defined as follows, and an illustrative example is provided at the end of this subsection.

Recall that $\mathscr{T}_H = \{\tau_1, ..., \tau_n\}$, where $\tau_1 < ... < \tau_n$. At the beginning of period τ_1, after the movements of units under policy IM (based on the rules described in §2.4.1), we assign indices from 1 to $y^B_{\tau_1}$ for units under policy B, and assign indices from 1 to $y^{IM}_{\tau_1}$ for units under policy IM,[3] where $y^B_{\tau_1} < y^{IM}_{\tau_1}$. Older units are assigned smaller indices, and units of the same age are assigned indices in an arbitrary sequence. Then, we *temporarily* match each unit ordered at τ_1 under policy B to the unit with the same index under policy IM. Clearly, each pair of temporarily matched units have the same age (of age 0). At the beginning of period τ_2, consider the following three cases.

Case 1: If a temporarily matched unit under policy B has been used to satisfy demand, there must exist a unit under policy IM that is also used to satisfy the same unit of demand. This is true because $y^B_{\tau_1} < y^{IM}_{\tau_1}$ and the inventory under policy IM is consumed no faster

[3] For continuous demands and ordering quantities, indices are defined continuously from 0 to $y^B_{\tau_1}$ and $y^{IM}_{\tau_1}$.

than that under policy B (due to Inequality (2.3)). Therefore, for a temporarily matched unit under policy B that has been used to satisfy demand, we re-match it to the unit under policy IM that is used to satisfy the same unit of demand, and we set this matching to be *permanent*.

Case 2: If a temporarily matched unit under policy B has been outdated, its last temporarily matched unit under policy IM must also have been outdated (because they have the same age). We set this matching also to be permanent.

Case 3: If a temporarily matched unit under policy B is still in inventory, we re-define the index and re-match it to a (potentially) new unit under policy IM. In particular, we assign indices from 1 to $y^B_{\tau_2}$ to units under policy B and assign indices from 1 to $y^{IM}_{\tau_2}$ to units under policy IM (after the movements of units). Then, we re-match (still temporarily) all previously temporarily matched units (now of age $\tau_2 - \tau_1$) and all units ordered at τ_2 (now of age 0) under policy B to units with the same indices under policy IM. Since there is only positive inventory of age 0 and $\tau_2 - \tau_1$ under policy IM and Equation (2.2) holds after the movements of units in period τ_2, each pair of temporarily matched units must have the same age (either 0 or $\tau_2 - \tau_1$).

Continuing in this manner, all units ordered in periods $t \in \mathscr{T}_H$ under policy B are ultimately permanently matched to certain units under policy IM if they are used to satisfy demand or outdated. For units that are ordered in $t \in \mathscr{T}_H$ and are still in inventory at the end of the planning horizon, their last temporarily matched units under policy IM must also be in inventory. This is because after the movements in $\tau_n \in \mathscr{T}_H$ (i.e., the last period in $\mathscr{T}_H$), Inequality (2.3) holds, which ensures that the inventory under policy IM is consumed no faster than that under policy B. We then set these matchings also to be permanent. Note that by construction, there are no overlaps in the permanent matchings. Further, as we will show in Lemma 5, any matched unit under policy B stays in inventory no longer than its permanently matched unit under policy IM.

Next, we illustrate the dynamic unit-matching scheme in Figure 2.2 based on the ex-

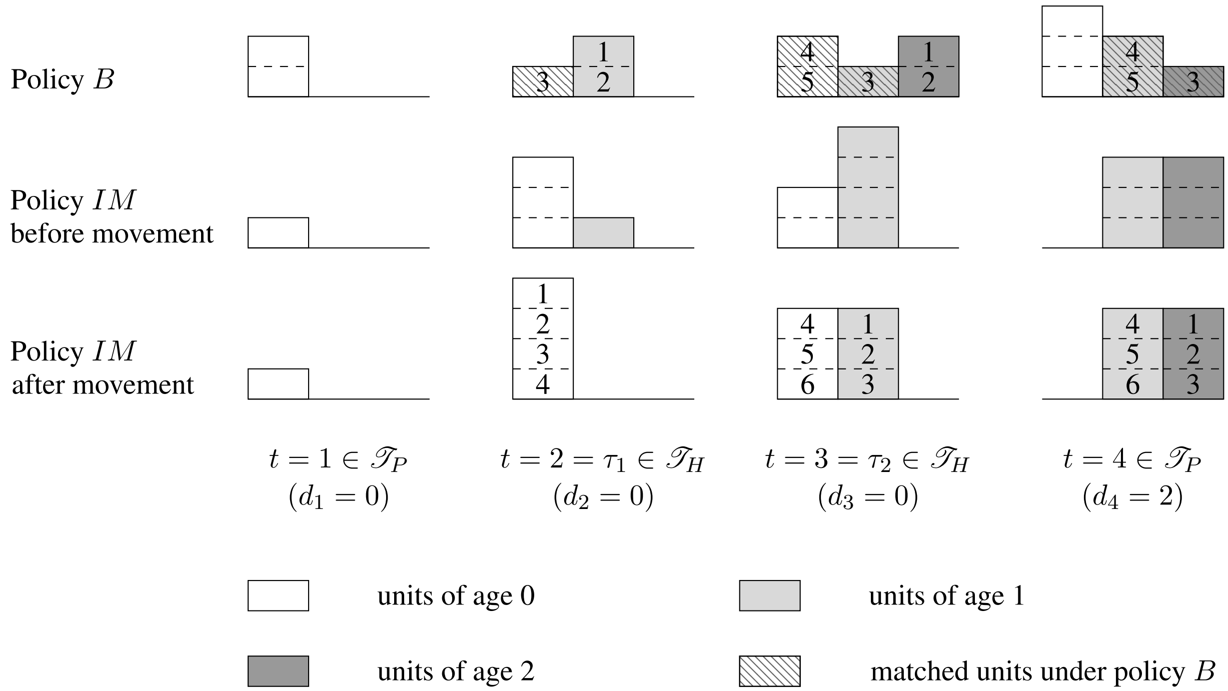


Figure 2.2: An illustrative example to show the dynamic unit-matching scheme

ample presented in §2.4.1. At the beginning of period $t = 1 \in \mathscr{T}_P$, no matching is defined. At the beginning of period $t = 2 = \tau_1$, after the movements under policy IM, units under policy B are assigned with indices from 1 to 3, units under policy IM are assigned with indices from 1 to 4, and unit 3 under policy B (ordered at τ_1) is temporarily matched to unit 3 under policy IM; units 1-2 are not matched since they are not ordered at periods in $\mathscr{T}_H$. At the beginning of period $t = 3 = \tau_2$, no permanent matching is defined since unit 3 under policy B is still in inventory. After the movements under policy IM, units under policy B are assigned with indices from 1 to 5, units under policy IM are assigned with indices from 1 to 6, and units 3-5 under policy B (ordered at either τ_1 or τ_2) are temporarily matched to units 3-5 under policy IM, respectively. At the beginning of period $t = 4 \in \mathscr{T}_P$, no new temporary matching is defined; also, since units 3-5 are all in inventory, no permanent matching is defined. At the end of the horizon (i.e., end of period 4), units 3-4 under policy B is permanently matched to units 1-2 under policy IM since they are used to satisfy the same units of demand. Unit 5 under policy B is still in inventory at the end of the horizon, therefore it is permanently matched to its last temporarily matched unit, i.e., unit 5 under policy IM.

2.4.3 Worst-Case Performance Guarantee.

We now prove the worst-case performance guarantee for policy B. As discussed before, we use policy IM as a bridging policy, and show that the expected total cost of policy IM is no more that that of policy OPT, and the expected total cost of policy B is at most twice that of policy IM, respectively.

Comparison of Policies IM and OPT. Recall that under Policy IM, an optimal ordering decision rule is implemented in each period while old products can be replaced with new ones for free. While is intuitive that replacing old products with new ones does not increase the expected total cost, it is not obvious when this is guaranteed to be true. In that regard, we find that FIFO being an optimal issuing policy is sufficient: FIFO being an optimal issuing policy implies that younger products are more preferred to have in inventory than older ones, thus replacing old products with new ones will not increase the expected total cost. We next formally describe the optimality of an issuing policy, followed by the statement of our assumption.

At each period t, given the inventory vector $\mathbf{x}_t$, the information set f_t and the ordering quantity q_t, let $v_{k,t}$ be the amount of products of age k that are used to meet demand at period $t, k = 0, ..., K-1$. Then, $v_{k,t} \leq x_{k,t}, k = 0, ..., K-1$, and $\sum_{k=0}^{K-1} v_{k,t} = \min\{y_t, d_t\}$. An issuing *decision rule* is a function from the set of all possible $(\mathbf{x}_t, f_t, q_t, d_t)$ to the set of all possible $\mathbf{v}_t = (v_{0,t}, ..., v_{K-1,t})$; and an issuing *policy* is a collection of issuing decision rules at all periods. Then, the FIFO issuing policy is such that at each period t, $v_{k,t} = \min\{x_{k,t}, (d_t - \sum_{m=k+1}^{K-1} x_{m,t})^+\}, k = 0, ..., K-1$. Given an initial inventory level and an ordering policy, an issuing policy is said to be optimal if it minimizes the expected total cost among all issuing policies. Let Φ_t be the cumulative distribution function (c.d.f.) of D_t for given f_t, and let the inverse c.d.f. of D_t be $\Phi_t^{-1}(z) := \inf\{x : \Phi_t(x) \geq z\}$. Given f_t, define the critical fractile at period t as $\tilde{y}_t(f_t) := \Phi_t^{-1}(\frac{p}{p+h})$, and define $\bar{y}_t := \max_{1\le s\le t} \sup_{f_s} \tilde{y}_s(f_s)$. Then, as we show in the proof of Lemma 3, the total inventory level after ordering at period t under an optimal policy is at most $\bar{y}_t$. To compare policies IM and OPT, we state our assumption as follows:

Assumption 1 (FIFO is Optimal) *At any period t, if $y_t \le \bar{y}_t$ and an optimal ordering decision rule (defined in §2.2 when the issuing policy is fixed as FIFO) is implemented at $t+1, ..., T$, then for any demand realization $d_t \ge 0$ and future demand distribution defined by any $f_{t+1} \in \mathscr{F}_{t+1}$, FIFO minimizes the expected total discounted cost in periods $t, ..., T$ among all issuing policies.*

Remark 1 *We note that we only assume FIFO to be optimal for relatively small initial inventory levels (i.e., $y_t \le \bar{y}_t$), which is a weaker assumption than assuming FIFO to be optimal under any initial inventory levels ([7]). This is because when the initial inventory level is high, FIFO may result in a higher holding cost without reducing much shortages or outdates compared with other issuing policies, in which case FIFO is less likely to be optimal.*

To date, FIFO has been shown to be optimal under i.i.d. demand ([6]) or zero holding cost ([7]). In §2.6, we further extend these existing findings and present a necessary and sufficient condition and a easy-to-check sufficient condition to ensure the optimality of the FIFO issuing policy. We also provide conditions and commonly seen examples under which FIFO is optimal and our performance guarantee is strictly tighter than the existing one.

With Assumption 1, we now present a structural property on the optimal cost-to-go function, which is a key result for comparing policies IM and OPT. At each period t, given $\mathbf{x}_t$ and f_t, let $C_t(\mathbf{x}_t, f_t)$ be the optimal cost-to-go function. Then, we have:

$$C_t(\mathbf{x}_t, f_t) = \min_{q_t \geq 0} \Big\{ p\mathrm{E}[(D_t - y_t)^+ | f_t] + h\mathrm{E}[(y_t - D_t)^+ | f_t] + w\mathrm{E}[(x_{K-1,t} - D_t)^+ | f_t] \\ + \beta\mathrm{E}[C_{t+1}(\mathbf{X}_{t+1}, F_{t+1})] \Big\}.$$

For $k = 1, ..., K-1$, for the continuous case, let $C_t^{(k)}(\mathbf{x}_t, f_t)$ denote the right partial derivative of $C_t(\mathbf{x}_t, f_t)$ with respect to to $x_{k,t}$ (i.e., $\partial C_t(\mathbf{x}_t, f_t)/\partial x_{k,t}$; the differentiability can be easily established following similar arguments as in [6]); for the discrete case, let $C_t^{(k)}(\mathbf{x}_t, f_t)$ denote the incremental of $C_t(\mathbf{x}_t, f_t)$ caused by a unit increase of $x_{k,t}$. Then, we have the following result.

Lemma 3 *Under Assumption 1, for* $t = 1, ..., T$, *(i)* $C_{t+1}^{(k)}(\boldsymbol{x}_{t+1}, f_{t+1}) \geq 0, k = 1, ..., K-1$, $\forall \boldsymbol{x}_{t+1}, f_{t+1}$; *(ii)* $C_{t+1}^{(i)}(\boldsymbol{x}_{t+1}, f_{t+1}) \leq C_{t+1}^{(j)}(\boldsymbol{x}_{t+1}, f_{t+1}), 1 \leq i < j \leq K-1$, $\forall \boldsymbol{x}_{t+1}$ *such that* $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$; *(iii)* $C_{t+1}^{(k)}(\boldsymbol{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1$, $\forall \boldsymbol{x}_{t+1}$ *such that* $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$.

Lemma 3 implies that when FIFO is an optimal issuing policy (i.e., Assumption 1), the optimal cost-to-go function is non-decreasing in the inventory levels; and if the total inventory level is small ($\leq \bar{y}_t$), the incremental of the optimal cost-to-go caused by an increase of an older unit is higher than that caused by a younger unit, and both are bounded by w/β. Then, since units are only moved from older to younger positions under policy IM, it is intuitive that the expected total cost of policy IM is no more than that of policy OPT, which leads to the following result.

Lemma 4 *Under Assumption 1,* $\mathrm{E}[\mathscr{C}(IM)] \leq \mathrm{E}[\mathscr{C}(OPT)]$.

Comparison of Policies B and IM. With Lemma 4, to establish the worst-case performance guarantee of policy B, it remains to show that $\mathrm{E}[\mathscr{C}(B)] \leq 2\mathrm{E}[\mathscr{C}(IM)]$. To compare the expected total cost under policies B and IM, we first provide a key lemma as follows.

Lemma 5 *With probability one, (i)* $\sum_{t \in \mathscr{T}_P} P_t^B \leq \sum_{t=1}^{T} P_t^{IM}$*; (ii)* $\sum_{t \in \mathscr{T}_H} H_t^B \leq \sum_{t=1}^{T} H_t^{IM}$*; and (iii)* $\sum_{t \in \mathscr{T}_H} W_t^B \leq \sum_{t=1}^{T} W_t^{IM}$.

Proof. (i) For any given sample path f_{T+1}, after the movements of units under policy IM, we have $y_t^B \geq y_t^{IM}, \forall t \in \mathscr{T}_P$. Then, $P_t^B \leq P_t^{IM}, \forall t \in \mathscr{T}_P$. Therefore, $\sum_{t \in \mathscr{T}_P} P_t^B \leq \sum_{t \in \mathscr{T}_P} P_t^{IM} \leq \sum_{t=1}^{T} P_t^{IM}$ with probability one.

(ii) To show that the total holding cost charged for units ordered at $t \in \mathscr{T}_H$ under policy B is no more than the total holding cost under policy IM, it is sufficient to show that under the dynamic unit-matching scheme described in §2.4.2, all matched units under policy B stay in inventory no longer than their permanently matched units under policy IM. Consider the following two cases:

Case 1: If a unit ordered at $t \in \mathscr{T}_H$ under policy B is used to satisfy demand (call it u_1), then by construction, its permanently matched unit under policy IM is used to satisfy the same unit of demand. Consider a temporarily matched pair of units, by Inequality (2.3), the unit under policy IM is consumed no earlier than its matched unit under policy B. Therefore, u_1's last temporarily matched unit under policy IM, which has the same index and age as u_1, is consumed no earlier than u_1. Then, given the FIFO issuing policy, u_1's permanently matched unit, which is used to satisfy the same unit of demand as u_1, must be no younger than u_1.

Case 2: If a unit ordered at $t \in \mathscr{T}_H$ under policy B is outdated or still in inventory at the end of the planning horizon (call it u_2), then its permanently matched unit under policy IM is defined as its last temporarily matched unit. Since each temporarily matched pair of units have the same age, and further considering possible movements from older to younger positions for units under policy IM, u_2 must have stayed in inventory no longer than its permanently matched unit.

Since the permanent matchings are defined on a one-to-one correspondence and the above arguments are true for any given sample path, we have $\sum_{t \in \mathscr{T}_H} H_t^B \leq \sum_{t=1}^{T} H_t^{IM}$ with probability one.

(iii) By the dynamic unit-matching scheme, for a matched unit under policy B that is outdated, the permanently matched unit under policy IM must be outdated at the same period. Since the permanent matchings are defined on a one-to-one correspondence and the above argument is true for any given sample path, we have $\sum_{t \in \mathscr{T}_H} W_t^B \le \sum_{t=1}^{T} W_t^{IM}$ with probability one. $\square$

With the above result, now it is easy to reach to the following conclusion.

Lemma 6 $\mathrm{E}[\mathscr{C}(B)] \le 2\mathrm{E}[\mathscr{C}(IM)]$.

Proof. Let $\mathbb{1}(t \in \mathscr{T}_P)$ and $\mathbb{1}(t \in \mathscr{T}_H)$ be two indicator functions. Then, we have $\mathbb{1}(t \in \mathscr{T}_P) + \mathbb{1}(t \in \mathscr{T}_H) = 1$ with probability one, and we have the following result.

$$
\begin{aligned}
\mathrm{E}[\mathscr{C}(B)] =& \sum_{t=1}^{T} \mathrm{E}[\mathrm{E}[P_t^B + H_t^B + W_t^B | F_t]] \\
=& \sum_{t=1}^{T} \mathrm{E}[\mathrm{E}[(P_t^B + H_t^B + W_t^B)(\mathbb{1}(t \in \mathscr{T}_P) + \mathbb{1}(t \in \mathscr{T}_H)) | F_t]] \\
=& \sum_{t=1}^{T} \mathrm{E}[\mathrm{E}[2P_t^B \mathbb{1}(t \in \mathscr{T}_P) + 2(W_t^B + H_t^B)\mathbb{1}(t \in \mathscr{T}_H) | F_t]] \\
=& \mathrm{E}\Bigg[\sum_{t \in \mathscr{T}_P} 2P_t^B + \sum_{t \in \mathscr{T}_H} 2(H_t^B + W_t^B)\Bigg] \\
\le& \mathrm{E}\Bigg[2\sum_{t=1}^{T} P_t^{IM} + 2\sum_{t=1}^{T} (H_t^{IM} + W_t^{IM})\Bigg] \\
=& 2\mathrm{E}[\mathscr{C}(IM)],
\end{aligned}
$$

where the third equality follows from the definition of the balancing policy and the fact that $\mathbb{1}(t \in \mathscr{T}_P), \mathbb{1}(t \in \mathscr{T}_H)$ are deterministic for given f_t; and the inequality follows from Lemma 5. $\square$

Based on the above results, we now state our main theorem as follows.

Theorem 1 *Under Assumption 1, the marginal-cost dual-balancing policy* (B) *has a worst-case performance guarantee of two. That is, the expected total cost of the marginal-cost* *dual-balancing policy is at most twice that of an optimal ordering policy, i.e.,* $\mathrm{E}[\mathscr{C}(B)] \leq 2\mathrm{E}[\mathscr{C}(OPT)]$.

Proof. Since we have $\mathrm{E}[\mathscr{C}(IM)] \leq \mathrm{E}[\mathscr{C}(OPT)]$ from Lemma 4 and $\mathrm{E}[\mathscr{C}(B)] \leq 2\mathrm{E}[\mathscr{C}(IM)]$ from Lemma 6, we have $\mathrm{E}[\mathscr{C}(B)] \leq 2\mathrm{E}[\mathscr{C}(IM)] \leq 2\mathrm{E}[\mathscr{C}(OPT)]$, which completes the proof. □

2.4.4 A Tight Example.

We now present an example to show that the performance guarantee of two is tight. Consider an instance with product lifetime $K = 1$, planning horizon $T = 1$, and cost parameters $p \gg 1, h = 1$ and $w = 0$. Assume demand is such that $\mathrm{P}(D_1 = 1) = \frac{1}{2}, \mathrm{P}(D_1 = 0) = \frac{1}{2}$.

Then, under the balancing policy, the ordering quantity at $t = 1$ (i.e., q_1^B) is determined by:

$$\frac{1}{2}p(1 - q_1^B) = \frac{1}{2}q_1^B.$$

Hence, $q_1^B = \frac{p}{1+p}$, and the expected total cost under policy B is $2 \times \frac{1}{2}q_1^B = \frac{p}{1+p}$.

Consider an alternative policy A whose ordering quantity at $t = 1$ is $q_1^A = 1$. Clearly, there is no shortage under policy A, and hence the expected total cost under policy A is $\frac{1}{2}$.

Therefore, when $p \to \infty$, we have $\frac{\mathrm{E}[\mathscr{C}(B)]}{\mathrm{E}[\mathscr{C}(OPT)]} \geq \frac{\mathrm{E}[\mathscr{C}(B)]}{\mathrm{E}[\mathscr{C}(A)]} \to 2$.

2.4.5 A Counterexample When FIFO Is Not Optimal.

We next present a counterexample to show that when FIFO is not optimal, the expected total cost of the balancing policy can be strictly more than twice that of an optimal policy. Consider an instance with product lifetime $K = 3$, planning horizon $T = 5$, cost parameters $p \gg 1, h = 1, w = 0$, and discount factor $\beta = 1$. Let $\epsilon > 0$ be a small number. Assume that demand across different periods is independent, and $D_1 = 0, \mathrm{P}(D_2 = 0) = 1 - \epsilon, \mathrm{P}(D_2 = 1) = \epsilon, \mathrm{P}(D_3 = 0) = 1 - \epsilon, \mathrm{P}(D_3 = 2) = \epsilon, D_4 = 1$ and $D_5 = 0$.

Then, under the balancing policy, the ordering quantity at $t = 1$ is clearly $q_1^B = 0$, and the ordering quantity at $t = 2$ (i.e., q_2^B) is determined by solving:

$$\epsilon p(1 - q_2^B) = (1 - \epsilon)q_2^B + (1 - \epsilon)^2 q_2^B.$$

Suppose the realized demand at $t = 2$ is $d_2 = 0$ (with probability $1-\epsilon$). Then, the balancing ordering quantity at $t = 3$ (denoted as $q_{3,d_2=0}^B$) is determined by solving:

$$\epsilon p(2 - q_2^B - q_{3,d_2=0}^B) = (1 - \epsilon)q_{3,d_2=0}^B + (1 - \epsilon) \times 2 \times (q_2^B + q_{3,d_2=0}^B - 1).$$

Suppose the realized demand at $t = 2$ is $d_2 = 1$ (with probability ϵ). Then, the balancing ordering quantity at $t = 3$ (denoted as $q_{3,d_2=1}^B$) is determined by solving:

$$\epsilon p(2 - q_{3,d_2=1}^B) = (1 - \epsilon)q_{3,d_2=1}^B + (1 - \epsilon) \times 2 \times (q_{3,d_2=1}^B - 1).$$

Let $p \to \infty$. Then $q_2^B \to 1, q_{3,d_2=0}^B \to 1, q_{3,d_2=1}^B \to 2$, and the expected total cost under policy B:

$$\mathrm{E}[\mathscr{C}(B)] \to 2(1-\epsilon)+2(1-\epsilon)^2+2(1-\epsilon)\big(1-\epsilon+2(1-\epsilon)\big)+2\epsilon\big(2(1-\epsilon)+2(1-\epsilon)\big) = 10+o(\epsilon),$$

where we use $o(\epsilon)$ to denote a small quantity that goes to zero when ϵ goes to zero.

Consider an alternative policy A whose ordering quantities at different periods are $q_1^A = 1, q_2^A = 0, q_{3,d_2=0}^A = 1, q_{3,d_2=1}^A = 2$, and $q_4^A = q_5^A = 0$. Clearly, there is no shortage under policy A, and it is not difficult to show that the expected total cost under policy A is $\mathrm{E}[\mathscr{C}(A)] = 4 + o(\epsilon)$.

Therefore, when ϵ is sufficiently small, we have $\frac{\mathrm{E}[\mathscr{C}(B)]}{\mathrm{E}[\mathscr{C}(OPT)]} \geq \frac{\mathrm{E}[\mathscr{C}(B)]}{\mathrm{E}[\mathscr{C}(A)]} = \frac{10+o(\epsilon)}{4+o(\epsilon)} > 2.$

2.5 Truncated-Balancing Policy.

In this section, we first prove that a myopic policy under the marginal-cost accounting scheme provides a lower bound on the optimal ordering quantity at each period. Then, by combining the *specific* lower bound we derive and any upper bound on the optimal ordering quantity with the balancing policy, we present a more general class of algorithms that we call the truncated-balancing policy. We later show that while both the balancing policy and the truncated-balancing policy have the same worst-case performance guarantee of two, the latter performs much better in the computational studies (see §4.7 for details).

In the following proposition, we show that when FIFO is optimal, the minimizer of the expected total marginal costs provides a lower bound on the optimal ordering quantity. We note that while a similar result has been developed in [25] for the nonperishable backlogging case (where the total inventory level is known to be a sufficient statistic for the system state), generalizing this result to the lost sales case (where a pipeline inventory vector is needed to describe the system state) remains an open problem. Our problem is similar to the lost sales case in the sense that we also need an inventory vector to describe the system state, and the analysis for the nonperishable backlogging case is not applicable to our case.

At any period t, given $\mathbf{x}_t$ and f_t, let q_t^{OPT} and q_t^L be an optimal ordering quantity and the smallest quantity that minimizes $P_t(\mathbf{x}_t, f_t, q_t) + H_t(\mathbf{x}_t, f_t, q_t) + W_t(\mathbf{x}_t, f_t, q_t)$, respectively. Then:

Proposition 1 *Under Assumption 1, $q_t^L \leq q_t^{OPT}$.*

Remark 2 *The key reason why a myopic ordering quantity under the marginal-cost accounting scheme provides a lower bound on the optimal ordering quantity is that, under the marginal-cost accounting scheme, the optimal cost-to-go function is monotonically non-increasing in the inventory levels of all ages. Thus, ordering more units can decrease the optimal cost-to-go of the next period, and the minimizer of the single-period cost, which ignores the benefit of ordering more for future, tends to order less than optimal. We remark* *that while this monotonicity result for the optimal cost-to-go function is easily established for the nonperishable inventory case, it can in fact be violated for the perishable inventory case in general. However, as we show in the proof of Proposition 1, the optimality of FIFO issuing policy is a sufficient condition to establish the monotonicity result, which we believe is a new and important contribution to the literature.*

Based on the above result, we next define the *truncated-balancing policy* (denoted as TB) as follows. At each period t, given $\mathbf{x}_t$ and f_t, let q_t^B be the balancing ordering quantity defined by Equation (2.1), let q_t^L be the lower bound on the optimal ordering quantity defined in Proposition 1 (or any *looser* lower bound), and let q_t^U be any upper bound on the optimal ordering quantity. Then, the truncated-balancing ordering quantity q_t^{TB} is defined as:

$$q_t^{TB} = \begin{cases} q_t^B, & \text{if } q_t^L \le q_t^B \le q_t^U, \\ q_t^L, & \text{if } q_t^B < q_t^L, \\ q_t^U, & \text{if } q_t^B > q_t^U. \end{cases}$$

Theorem 2 *Under Assumption 1, the truncated-balancing policy (TB) has a worst-case performance guarantee of two. That is, the expected total cost of the truncated-balancing policy is at most twice that of an optimal ordering policy, i.e.,* $\mathrm{E}[\mathscr{C}(TB)] \le 2\mathrm{E}[\mathscr{C}(OPT)]$.

Remark 3 *We remark that policy TB is not guaranteed to perform at least as good as policy B (thus the proof of Theorem 2 is nontrivial). While it may appear that q_t^{TB} is at least as good as q_t^B, this is only true if an optimal policy is implemented at the following periods.*

Remark 4 *We also remark that unlike the nonperishable inventory case where the lower and upper bounds in the definition of policy TB can be replaced with any (tighter) ones, in our case, the lower bound q_t^L, as a minimizer of the single-period marginal cost, is special and cannot be tightened. To see why this is the case, let y_t^{TB} and y_t^{IM} be the total inventory levels after ordering at period t under policies TB and IM, respectively; also,* *given $\boldsymbol{x}_t^{TB}$ and f_t, let y_t^B denote the total inventory level after ordering if the balancing ordering quantity q_t^B is ordered. Suppose $q_t^B < q_t^{TB} = q_t^L$. Then, it is possible that $y_t^B \leq y_t^{IM} < y_t^{TB}$ (while for the nonperishable case, since a base-stock policy is optimal, given that $q_t^B < q_t^{TB} = q_t^L$, we always have $y_t^B < y_t^{TB} = q_t^L \leq y_t^{OPT}$). In this case, since $y_t^{IM} < y_t^{TB}$, it is not possible to match all the q_t^{TB} units to units under policy IM. Therefore, we instead only match the first q_t^B units ordered under policy TB to units under policy IM, and then show that the expected total marginal cost at period t for ordering q_t^{TB} is no more than that for ordering q_t^B, which is only ensured to be true if $q_t^B < q_t^{TB} \leq$ the minimizer of the single-period expected marginal cost (see more details in the proof of Theorem 2 in the appendix).*

2.6 Sufficient Conditions for Optimality of FIFO Issuing Policy.

In §2.4-2.5, we have shown that our proposed algorithms have a worst-case performance guarantee of two when FIFO is an optimal issuing policy (i.e., Assumption 1). In this section, we provide a necessary and sufficient condition and a easy-to-check sufficient condition that ensure the optimality of FIFO issuing policy, which extends the existing literature and provides insights into the key trade-offs of different issuing policies.

In Lemma 3, we have presented a necessary condition for the optimality of FIFO issuing policy. In the following proposition, we show that this condition is also sufficient (in fact, we only need a part of that condition), which leads to a *necessary and sufficient* condition for the optimality of FIFO issuing policy.

Proposition 2 *Assumption 1 holds if and only if for $t = 1, ..., T$, $C_{t+1}^{(k)}(\boldsymbol{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1$, $\forall \boldsymbol{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$.*

Proposition 2 says that FIFO is optimal if and only if the incremental of the discounted optimal cost-to-go caused by a unit increase of the inventory of any age is bounded by the unit outdating cost. This provides an overall insight into the key trade-off in ensuring the optimality of FIFO issuing policy. In particular, a unit increase of inventory can potentially increase both holding and outdating costs but decrease the shortage penalty. Consider a case where demand for future periods is sufficiently large so that the decrease in shortage penalty offsets the increase in holding cost. In this case, the incremental of the total cost is bounded by the unit outdating cost, i.e., the condition in Proposition 2 holds. Also, if the unit outdating cost is sufficiently large, then the total discounted holding and outdating costs for future periods may also be bounded by the unit outdating cost, regardless of the demand distribution. In this case, the condition in Proposition 2 also holds.

Based on the above intuition, we next present an easy-to-check sufficient condition that ensures the optimality of FIFO issuing policy. This condition involves both cost parameters and demand distributions, and extends the existing findings on the optimality of FIFO issuing policy ([6, 7]). We describe in further details below.

Proposition 3 *If* $h \leq \frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$, *where* $\gamma = \sup_{f_{T+1}} \max_{1 \leq t \leq T} \Phi_t(\bar{y}_t)$, *then Assumption 1 holds.*

We believe this is an authentic result and provides key insights into the main trade-offs of FIFO issuing policy. In particular, Proposition 3 says that for fixed shortage penalty p, holding cost h and discount factor β, FIFO is optimal when demand over time does not "drop" significantly (so that γ is not too large) or the unit outdating cost is large. Moreover, the larger the unit outdating cost, the weaker the requirement we need for the demand, and vice versa.

We also remark that the condition in Proposition 3 does not necessarily require the holding cost h to be small. This is because when h increases, the ratio $\frac{p}{p+h}$ would decrease, which decreases the value of γ and increases the value of $\frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$. Indeed, as we discuss in the end of this section, the inequality $h \leq \frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$ may be more likely to hold when h increases.

In the following corollary, we show that FIFO is ensured to be optimal when demand is independent and the critical fractile is non-decreasing in time, which extends the i.i.d.

condition considered in [11]. We also note that this condition is more general than the independent and stochastically non-decreasing demand (in the notion of first-order dominance) in [32].

Corollary 1 *If demand across different periods is independent and $\tilde{y}_t = \Phi_t^{-1}(\frac{p}{p+h})$ is non-decreasing in t, then Assumption 1 holds.*

Given Proposition 3, the proof for Corollary 1 is straightforward since if demand across different periods is independent and $\tilde{y}_t = \Phi_t^{-1}(\frac{p}{p+h})$ is non-decreasing in t, then $\bar{y}_t = \tilde{y}_t$ and $\gamma = \sup_{f_{T+1}} \max_{1\leq t\leq T} \Phi_t(\bar{y}_t) = \max_{1\leq t\leq T} \Phi_t(\bar{y}_t) = \max_{1\leq t\leq T} \Phi_t(\tilde{y}_t) = \frac{p}{p+h}$. Hence, $\frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w = h + \frac{1-\beta\gamma}{\beta\gamma}w \geq h$.

In the following corollary, we present another condition to ensure FIFO optimality which involves only cost parameters and does not impose any condition on demand distribution.

Corollary 2 *If $h \leq \frac{1-\beta}{\beta}w$, then Assumption 1 holds.*

The proof for Corollary 2 is also straightforward. Since $\frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$ is non-increasing in γ and we always have $\gamma \leq 1$ as γ represents a probability, we have $h \leq \frac{1-\beta}{\beta}w \leq \frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$. We also note that the condition in Corollary 2 to ensure the optimality of FIFO issuing policy extends the zero holding cost assumption considered in [7].

We also note that the condition $h \leq \frac{1-\beta}{\beta}w$ is intuitive and implies that the total discounted holding cost for an arbitrary number of future periods $(\beta + \beta^2 + ...)h = \beta h/(1-\beta)$ is bounded by a unit outdating cost w at the current period. Intuitively, this condition ensures the optimality of FIFO because if holding a unit for ever is less expensive than letting a unit outdate, then one would never issue a young product while letting an old product outdate.

In the remainder of this section, we present two scenarios corresponding to zero and positive (original) holding cost, respectively, and illustrate the conditions we identified in this section enables us to tighten the existing worst-case performance guarantee in both scenarios.

Zero (Original) Holding Cost. Small or zero (*original*, instead of transformed) holding cost is commonly seen in the perishable inventory literature since the main concern for over-ordering in perishable inventory systems is outdating [35, 36]. We first consider an instance with zero holding cost $\hat{h} = 0$. Then, for any other cost parameters such that $\hat{p} \geq \hat{c} \geq 0$ and $\hat{w} \geq 0$, the transformed cost parameters are $c = 0, p = \hat{p} - \hat{c} \geq 0, h = \hat{h} + (1-\beta)\hat{c} = (1-\beta)\hat{c}$ and $w = \hat{w} + \beta\hat{c} \geq \beta\hat{c}$. Note that although the original holding cost is $\hat{h} = 0$, the transformed holding cost h is strictly positive (as long as $\beta < 1$). However, since $h = (1-\beta)\hat{c} = \frac{1-\beta}{\beta}\beta\hat{c} \leq \frac{1-\beta}{\beta}w$, Assumption 1 holds under general demand (Corollary 2) and hence the worst-case bound of our algorithms is two.

On the other hand, under general demand, the performance guarantee presented in Chao et al. [32] is $2 + \frac{(K-2)h}{Kh+w} = 2 + \frac{(K-2)(\hat{h}+(1-\beta)\hat{c})}{K(\hat{h}+(1-\beta)\hat{c})+\hat{w}+\beta\hat{c}}$. We set $K = 5, \hat{c} = 1, \hat{w} = 0$, and present this existing worst-case bound and our bound under different values of discount factor in Table 2.2.

Table 2.2: Comparison between existing and our worst-case bound under zero holding cost.

β	0.8	0.85	0.9	0.95	0.99	1
γ	≤ 1	≤ 1	≤ 1	≤ 1	≤ 1	≤ 1
h	0.2	0.15	0.1	0.05	0.01	0
$\frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$	≥ 0.2	≥ 0.15	≥ 0.1	≥ 0.05	≥ 0.01	0
Existing bound	2.33	2.28	2.21	2.13	2.03	2
Our bound	2	2	2	2	2	2

We remark that for the existing performance bound presented by Chao et al. [32] to be exactly two, the *transformed* holding cost $h = \hat{h} + (1-\beta)\hat{c}$ needs to be zero, which requires not only the original holding cost $\hat{h}$ but also the original ordering cost $\hat{c}$ to be zero (under general lifetime K and discount factor β). In contrast, we only require $\hat{h}$ to be small (our condition $h \leq \frac{1-\beta}{\beta}w$ is equivalent to $\hat{h} \leq \frac{1-\beta}{\beta}\hat{w}$) and we allow the original ordering cost $\hat{c}$ to be arbitrarily large.

Positive Holding Cost. We next consider scenarios with positive holding costs. Specifically, we consider product lifetime $K = 5$ and original cost parameters $\hat{c} = 1$ and $\hat{h} = \hat{p} = \hat{w} = 5$. Then, the transformed cost parameters are $c = 0, p = \hat{p} - \hat{c} = 4, h = \hat{h} + (1-\beta)\hat{c} = 5 - \beta$ and $w = \hat{w} + \beta\hat{c} = 5 + \beta$. We consider nonstationary demand that can be stochastically decreasing from one period to another. Specifically, we assume that demand at periods $1, 3, 5, ...$ is exponentially distributed with mean of μ_{max}, and demand at periods $2, 4, 6, ...$ is exponentially distributed with mean of μ_{min} (we note that our results do not rely on the specific pattern of periodicity or the independence/correlation of demand across periods; it is the gap between μ_{max} and μ_{min} that matters).

Then, we have $\Phi_1(x) \leq \Phi_2(x), \forall x \geq 0$ and $\Phi_1^{-1}(x) \geq \Phi_2^{-1}(x), \forall x \in [0,1]$, hence $\gamma = \Phi_2(\Phi_1^{-1}(\frac{p}{p+h}))$. We present the existing worst-case bound and our bound under different values of β and μ_{min} (where μ_{max} is fixed to be 10) in Table 2.3, where our bound being 2 suggests that the condition in Proposition 3 holds, while being "NA" suggests that the condition does not hold.

From the above results, we observe that the condition $h \leq \frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$ is satisfied under a wide range of model parameters even though we consider a fairly large holding cost and allow demand to be stochastically decreasing from one period to another. These results imply that we are able to tighten the existing worst-case bound for a large class of perishable inventory systems.

Also, as noted earlier, our condition does not necessarily require small holding cost because increasing h decreases the value of γ and hence increases $\frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$. To better see this, we consider $\mu_{max} = 10, \mu_{min} = 5$, and $\beta = 0.95$ (which corresponds to an "NA" case in Table 2.3) and present existing and our worst-case bounds under a large range of holding costs in Table 2.4.

Table 2.3: Comparison between existing and our worst-case bounds under positive holding cost.

β		0.8	0.85	0.9	0.95	0.99	1
$\mu_{max} = 10, \mu_{min} = 9$	γ	0.470	0.472	0.475	0.477	0.479	0.480
	h	5.20	5.15	5.10	5.05	5.01	5
	$\frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$	14.14	13.21	12.32	11.57	10.99	10.83
	Existing bound	2.49	2.49	2.49	2.49	2.48	2.48
	Our bound	2	2	2	2	2	2
$\mu_{max} = 10, \mu_{min} = 7$	γ	0.557	0.560	0.563	0.565	0.568	0.568
	h	5.20	5.15	5.10	5.05	5.01	5
	$\frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$	10.40	9.58	8.85	8.21	7.70	7.61
	Existing bound	2.49	2.49	2.49	2.49	2.48	2.48
	Our bound	2	2	2	2	2	2
$\mu_{max} = 10, \mu_{min} = 5$	γ	0.681	0.683	0.686	0.689	0.691	0.691
	h	5.20	5.15	5.10	5.05	5.01	5
	$\frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$	6.72	6.08	5.49	4.95	4.55	4.47
	Existing bound	2.49	2.49	2.49	2.49	2.48	2.48
	Our bound	2	2	2	NA	NA	NA

2.7 Computational Experiments.

We start with a discussion on the computation of the balancing ordering quantity in §2.7.1. Then, in §2.7.2-2.7.3, we test the performances of our proposed policies under different demand distributions based on both hypothetical and real data.

2.7.1 Computation of Balancing Ordering Quantity.

At each period t, given $\mathbf{x}_t$ and f_t, the marginal shortage penalty $P_t(\mathbf{x}_t, f_t, q_t)$ is non-increasing in q_t, while the marginal holding and outdating costs $H_t(\mathbf{x}_t, f_t, q_t)$ and $W_t(\mathbf{x}_t, f_t, q_t)$ are non-decreasing in q_t. Thus the balancing ordering quantity q_t^B defined in Equation (2.1) can be computed using a simple binary search. However, to do so, we first need to efficiently compute the expected marginal costs for each given q_t. Given the distribution of D_t, the computation of $P_t(\mathbf{x}_t, f_t, q_t)$ is straightforward. Thus, in this subsection, we focus on the computation of $H_t(\mathbf{x}_t, f_t, q_t)$ and $W_t(\mathbf{x}_t, f_t, q_t)$.

Table 2.4: Comparison between existing and our worst-case bounds under a large range of holding costs.

$\hat{h}$		0	5	10	20	50	100
$\mu_{max} = 10, \mu_{min} = 5$	γ	1.000	0.689	0.488	0.305	0.143	0.075
	h	0.05	5.05	10.05	20.05	50.05	100.05
	$\frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$	0.31	4.95	11.08	23.70	61.82	126.89
	Existing bound	2.02	2.49	2.54	2.57	2.59	2.59
	Our bound	2	NA	2	2	2	2

Similar as the existing studies on balancing policies, for general demands, the expected marginal holding and outdating costs $H_t(\mathbf{x}_t, f_t, q_t)$ and $W_t(\mathbf{x}_t, f_t, q_t)$ can be computed using methods such as Monte Carlo simulation. However, if demand over time is independent and integer-valued (we consider integer-valued quantities in our computational experiments), we can further achieve closed-form expressions for the expected marginal costs as follows.

Recall that $A_{0,t} = 0$, and for $k = 1, ..., K-1$, $A_{k,t}$ denotes the total demand over periods $t, ..., t+k-1$ that cannot be satisfied by the inventory of $(x_{K-k,t}, ..., x_{K-1,t})$, and $A_{0,t} = 0$. Similar to [12], for given $(x_{K-k+1,t}, ..., x_{K-1,t})$ and f_t, define:

$$R_{k,t}(x_{K-k,t}) = \mathrm{P}(A_{k-1,t} + D_{t+k-1} < x_{K-k,t} | f_t), k = 1, ..., K,$$

which denotes the conditional probability that there will be outdates at the end of period

$t + k - 1$. Then, $\forall u \geq 0$, we have $R_{1,t}(u) = \mathrm{P}(D_t < u | f_t)$, and for $k = 2, ..., K$:

$$\begin{aligned}
R_{k,t}(u) &= \mathrm{P}(A_{k-1,t} + D_{t+k-1} < u | f_t) \\
&= \sum_{v=1}^{u} \mathrm{P}(A_{k-1,t} < v | f_t) \mathrm{P}(D_{t+k-1} = u - v | f_t) \\
&= \sum_{v=1}^{u} \mathrm{P}((A_{k-2,t} + D_{t+k-2} - x_{K-k+1,t})^+ < v | f_t) \mathrm{P}(D_{t+k-1} = u - v | f_t) \\
&= \sum_{v=1}^{u} \mathrm{P}(A_{k-2,t} + D_{t+k-2} < v + x_{K-k+1,t} | f_t) \mathrm{P}(D_{t+k-1} = u - v | f_t) \\
&= \sum_{v=1}^{u} R_{k-1,t}(v + x_{K-k+1,t}) \mathrm{P}(D_{t+k-1} = u - v | f_t).
\end{aligned}$$

Therefore, the probabilities $R_{k,t}$ can be computed efficiently by recursion. In this case, at each period t, given $\mathbf{x}_t$, f_t and q_t, the expected marginal holding and outdating costs can be computed as:

$$\begin{aligned}
H_t(\mathbf{x}_t, f_t, q_t) &= \sum_{k=0}^{K-1} h_{t+k} \mathrm{E}\left[(q_t - (A_{k,t} + D_{t+k} - \sum_{m=1}^{K-k-1} x_{m,t})^+)^+ \middle| f_t\right] \\
&= \sum_{k=0}^{K-1} h_{t+k} \sum_{u=1}^{q_t} \mathrm{P}\left((A_{k,t} + D_{t+k} - \sum_{m=1}^{K-k-1} x_{m,t})^+ < u \middle| f_t\right) \\
&= \sum_{k=0}^{K-1} h_{t+k} \sum_{u=1}^{q_t} \mathrm{P}\left(A_{k,t} + D_{t+k} < u + \sum_{m=1}^{K-k-1} x_{m,t} \middle| f_t\right) \\
&= \sum_{k=0}^{K-1} h_{t+k} \sum_{u=1}^{q_t} R_{k+1}\left(u + \sum_{m=1}^{K-k-1} x_{m,t}\right),
\end{aligned}$$

and

$$\begin{aligned}
W_t(\mathbf{x}_t, f_t, q_t) &= w_{t+K-1} \mathrm{E}[(q_t - A_{K-1,t} - D_{t+K-1})^+ | f_t] \\
&= w_{t+K-1} \sum_{u=1}^{q_t} \mathrm{P}(A_{K-1,t} + D_{t+K-1} < u | f_t) \\
&= w_{t+K-1} \sum_{u=1}^{q_t} R_{K,t}(u).
\end{aligned}$$

2.7.2 Experiments with Hypothetical Data.

We next numerically test the performance of the proposed policies under a wide range of data. In particular, we consider a) i.i.d. demand; b) periodic demand; and c) autoregressive demand in this subsection to capture both nonstationarity and correlation in the demand process (in addition to the i.i.d. case). Then, in §2.7.3, we consider a real data case based on platelet inventory management with dynamically evolving demand.

Independent and Identically Distributed (i.i.d.) Demand. We first consider i.i.d. demand, and similar to Cooper [17], we consider two discrete demand distributions, Poisson and geometric, to represent demand with small and large variances, respectively. We assume that the mean of demand is equal to 5. We consider zero ordering cost $\hat{c} = 0$,[4] fix the unit outdating cost to be $\hat{w} = 5$, and consider different combinations of unit shortage penalty and holding cost: $\hat{p} = 5, 10, 20$; $\hat{h} = 0, 1, 2$. Similar to many existing studies on perishable inventory systems [e.g. 13, 16, 17], we consider short product lifetimes $K = 2, 3$ in order to benchmark with an optimal policy. Finally, we consider a planning horizon $T = 20$ and a discount factor $\beta = 0.95$. We generate $10,000$ random scenarios to estimate the expected costs. Let $\bar{\mathscr{C}}(\pi)$ and $\bar{\mathscr{C}}(OPT)$ denote the estimated expected total cost under policies π and OPT, respectively. We define the performance error of policy π as:

$$error(\pi) := \frac{\bar{\mathscr{C}}(\pi) - \bar{\mathscr{C}}(OPT)}{\bar{\mathscr{C}}(OPT)} \times 100\%.$$

We report the performance error of policies PB and DB presented in [32] and our proposed policies B and TB[5] under i.i.d. demand in Table 2.5, where each number represents the performance error of each policy averaged across two different lifetimes and two different demand distributions. From these results, we first observe that the performance error of our proposed policies is much smaller than the worst-case performance bound of two (i.e.,

[4]The performance error of all policies becomes smaller with a positive $\hat{c}$ due to the resulting larger optimal cost.

[5]Policy TB here is truncated by the lower bound in Proposition 1 and is not truncated by any upper bound.

Table 2.5: Performance error of different policies under i.i.d. demand.

Policy		PB	DB	B	TB
$\hat{p} = 5$	$\hat{h} = 0$	1.5%	1.5%	1.5%	1.2%
	$\hat{h} = 1$	2.5%	2.5%	2.7%	1.1%
	$\hat{h} = 2$	2.9%	2.6%	2.9%	1.8%
$\hat{p} = 10$	$\hat{h} = 0$	3.3%	3.3%	3.3%	0.5%
	$\hat{h} = 1$	3.2%	3.2%	3.5%	0.4%
	$\hat{h} = 2$	2.6%	2.6%	2.9%	1.0%
$\hat{p} = 20$	$\hat{h} = 0$	3.5%	3.5%	3.5%	1.8%
	$\hat{h} = 1$	5.3%	5.9%	6.2%	1.8%
	$\hat{h} = 2$	3.3%	3.4%	3.9%	1.0%

an error of 100%). Second, the performances of policies PB, DB and B are similar to each other, which is expected as all these three policies tend to balance the overage and underage marginal costs (when $\hat{h} = 0$, these three policies become exactly the same). Finally, policy TB performs significantly better than all other policies, especially when the unit shortage penalty becomes large. This is because as we have shown in the tight example in §2.4.4, the balancing policy tends to under-order when the unit shortage penalty becomes large. In this case, the truncation by the lower bound we derive helps bring up the order quantity, which significantly reduces the performance error.

Periodic Demand. We next consider periodic demand, which has been widely considered for the management of perishable (especially blood) inventory [35, 36]. Same to the i.i.d. case, we consider Poisson and geometric demand. We now consider a weekly periodicity where the mean of demand on weekdays and weekends are equal to 10 and 5, respectively (as we have shown in §2.6, the sufficient condition we present to ensure the optimality of FIFO may not hold in this case). We consider the same cost parameters, horizon length and discount factor as the i.i.d. case, and report the performance error of different policies in Table 2.6. We now only benchmark with policy PB as policy DB is only well-defined for stochastically non-decreasing demand.

Similar to the i.i.d. case, we observe that both our policies B and TB perform much

Table 2.6: Performance error of different policies under periodic demand.

Policy		PB	B	TB
$\hat{p} = 5$	$\hat{h} = 0$	1.8%	1.8%	1.7%
	$\hat{h} = 1$	2.8%	2.9%	1.3%
	$\hat{h} = 2$	2.7%	2.5%	2.0%
$\hat{p} = 10$	$\hat{h} = 0$	2.0%	2.0%	0.9%
	$\hat{h} = 1$	3.6%	4.1%	0.7%
	$\hat{h} = 2$	2.5%	2.7%	0.8%
$\hat{p} = 20$	$\hat{h} = 0$	3.6%	3.6%	1.6%
	$\hat{h} = 1$	5.6%	6.5%	1.1%
	$\hat{h} = 2$	3.9%	4.6%	0.8%

better than the worst-case performance bound, and policy TB performs significantly better than other policies when the unit shortage penalty becomes large. Moreover, while FIFO may not be an optimal issuing policy when demand is nonstationary and holding cost is large (i.e., the worst-case performance guarantee of our policies may not hold), we observe that our policy TB continues to work better than existing policies even when the sufficient condition presented in §2.6 does not hold.

Autoregressive Demand. We next consider an autoregressive demand model, which has also been shown by real data to be a great fit for demand of blood products [37]. Specifically, we consider an AR(1) model with $D_t = D_{t-1} + \epsilon_t$, where $D_0 = 5$ and ϵ_t follows a normal distribution with mean of zero and variance of 5 and 30, respectively (to mimic to variances of Poisson and geometric distributions considered above). We further discretize the normal distribution and truncate any negative demand by zero. In order to benchmark with an optimal policy, we only consider product lifetime $K = 2$ under the AR(1) model, in which case the state space is augmented with the demand realization from the previous period. Other model parameters are considered the same as above, and the performance error of different policies are presented in Table 2.7.

We observe that under AR(1) demand, the performance error of policies PB and B are larger than in the independent cases. However, policy TB continues to perform substan-

Table 2.7: Performance error of different policies under AR(1) demand.

Policy		PB	B	TB
$\hat{p} = 5$	$\hat{h} = 0$	6.6%	6.6%	1.3%
	$\hat{h} = 1$	4.4%	4.4%	1.3%
	$\hat{h} = 2$	3.2%	3.2%	2.1%
$\hat{p} = 10$	$\hat{h} = 0$	10.2%	10.2%	1.9%
	$\hat{h} = 1$	7.1%	7.1%	1.0%
	$\hat{h} = 2$	4.9%	4.9%	1.0%
$\hat{p} = 20$	$\hat{h} = 0$	14.3%	14.3%	2.2%
	$\hat{h} = 1$	11.7%	11.7%	1.8%
	$\hat{h} = 2$	8.9%	8.9%	0.9%

tially better than other policies, especially when the unit shortage penalty becomes large.

2.7.3 Experiments Real Data for Platelet Inventory Management.

We next consider the platelet inventory management problem at Emory University Hospital Midtown (§4.1). At Emory Midtown, 1) platelets are ordered on a daily basis, and an order placed at the end of each day will arrive in the morning of the next day; 2) as demand arises, older platelets are typically issued first to reduce outdates; and 3) unmet demand is satisfied by emergency deliveries. Therefore, our assumptions for zero lead time, FIFO issuing policy, and lost sales are applicable in this setting.

Platelets have a short lifetime of $K = 3$ days, and we consider a planning horizon of 4 weeks (i.e., $T = 28$ days). As discussed in §4.1, we focus on the main source of demand for platelets: cardiac surgeries, and we model daily demand for platelets by a compound Poisson distribution ([3, 4, 5]). Similar to two recent studies by [35] and [36], we assume that demand over time is independently distributed, but the distribution in different days may not be identical. Based on the cardiac surgery records from January to April in 2014, we identify a significant weekly periodicity, and estimate the average number of surgeries from Monday to Sunday as 2.6, 5.5, 1.9, 3.2, 3.7, 0.1, and 0, respectively. We assume the amount of platelets needed per surgery is stationary; based on the platelet transfusion records, we estimate it as a geometric distribution with mean of 0.32.

Cardiac surgeries are usually scheduled days or even weeks in advance; therefore forecast information on the number surgeries per day is typically available. We consider a forecast horizon three days, and assume that the forecast is perfect. That is, the number of surgeries at day $t + 2$ becomes known at the beginning of day t, and will not change at the following days. In this case, at any day t, given f_t, each of the demand $D_t, ..., D_{t+K-1}$ becomes a sum of a deterministic number of i.i.d. geometric distributions (i.e., a negative binomial distribution) instead of a compound Poisson distribution.

Based on the interaction with the blood bank manager at Emory Midtown, we estimate the unit outdating cost w to be equal to the purchase cost $\$500$. On the other hand, the shortage penalty for blood inventory problems could include the cost of emergent shipment from other blood banks and/or the penalty of postponing the surgeries, which is usually high and often estimated as 2-10 times higher than the purchase cost ([35]). We consider three different shortage penalties $p = \$1000, \$2500, \$5000$ in this study. Also, we consider a zero holding cost $h = 0$ and no discount, i.e., $\beta = 1$ (in this case, policies B and PB become identical).

We first benchmark the performances of our policies with the optimal policy solved by dynamic programming. The state of the dynamic program is comprised of a $K - 1$ dimensional vector of inventory levels of age $1, ..., K - 1$, and a K dimensional vector of forecasts on the number of surgeries at days $t, ..., t + K - 1$. Although the problem size we face here is not too large, it still takes more than 50 hours to compute the optimal policy on a standard 2.6GHz PC, whereas the ordering quantities under our policies can be computed on the fly in an online fashion. On the other hand, while compound Poisson distribution is widely considered in the blood inventory literature (e.g., [3, 4, 5]), none of these studies has considered the forecast information on the number of patients per period. A natural question is that how much do we lose by ignoring this information? Therefore, we also compare the performances of our policies with the "optimal" policy that does not make use of the forecast information (i.e., it simply treats the demand at each day as a compound Poisson distribution and thus the state of this dynamic program is simply comprised of a $K-1$ dimensional vector of inventory levels of age $1, ..., K-1$).

We use OPT_{wof} to denote the "optimal" policy without forecast information, and characterize the value of forecast information by assessing the performance improvements of our policies over policy OPT_{wof}. Let $\bar{\mathscr{C}}(\pi)$ and $\bar{\mathscr{C}}(OPT_{wof})$ be the estimated total costs under policies π and OPT_{wof}, respectively. We define the performance improvement of policy π as:

$$impr(\pi) := \frac{\bar{\mathscr{C}}(OPT_{wof}) - \bar{\mathscr{C}}(\pi)}{\bar{\mathscr{C}}(OPT_{wof})} \times 100\%.$$

Table 2.8: Performance summary of different policies for the platelet inventory control problem.

Policy			PB/B	TB	OPT	OPT$_{wof}$
$p = \$1000$	$\bar{\mathscr{C}}$	(\$)	6813	6684	6174	7262
	$error$	(%)	10.4	8.3	0	17.6
	$impr$	(%)	6.2	8.0	15.0	0
$p = \$2500$	$\bar{\mathscr{C}}$	(\$)	10059	9666	8943	10532
	$error$	(%)	12.5	8.1	0	17.8
	$impr$	(%)	4.5	8.2	15.1	0
$p = \$5000$	$\bar{\mathscr{C}}$	(\$)	12689	11918	10990	12999
	$error$	(%)	15.5	8.4	0	18.3
	$impr$	(%)	2.4	8.3	15.5	0

The estimated total cost, performance error, and performance improvement of different policies are reported in Table 2.8. Similar to the hypothetical data cases, we first observe that both our policies B and TB perform much better than the worst-case performance bound. Further, policy TB performs significantly better than policy B, especially when the unit shortage penalty gets large. Finally, we also observe that policy OPT_{wof} performs poorly, whereas our policy TB has a substantial performance improvement (more than 8%) over policy OPT_{wof}. Therefore, the value of the forecast information is significant, and implementing an inventory policy that takes into account such information has a potential to achieve a much better performance in practice.

2.8 Conclusion.

In this paper, we consider a periodic-review fixed-lifetime perishable inventory management problem assuming that demand is a general stochastic process which can be nonstationary, correlated, and dynamically evolving. Theoretically, an optimal ordering policy of this problem can be solved using standard dynamic programming. However, it becomes computationally intractable for realistic size problems due to the high dimensionality of the state space. We first present a computationally efficient algorithm that we call the marginal-cost dual-balancing policy. We then prove that under the marginal-cost accounting scheme, the minimizer of the single-period cost provides a lower bound on the optimal ordering quantity. By combining the specific lower bound we derive and any upper bound on the optimal ordering quantity with the marginal-cost dual-balancing policy, we present a more general class of algorithms that we call the truncated-balancing policy. We prove that when FIFO is an optimal issuing policy, both of our policies have a worst-case performance guarantee of two, i.e., the expected total cost of our policies is at most twice that of an optimal policy. We further provide a necessary and sufficient condition and a easy-to-check sufficient condition that ensure the optimality of the FIFO issuing policy. Finally, we conduct extensive numerical analyses based on both hypothetical and real data and show that i) both of our policies perform significantly better than the theoretical worst-case performance guarantee, and ii) the truncated-balancing policy significantly outperforms the marginal-cost dual-balancing policy especially when the unit shortage penalty is large, which illustrates that the lower bound we derive is effective and helps to improve the performance of the balancing policy.

Our worst-case analysis is built on two novel ideas, the imaginary operation policy and the dynamic unit-matching scheme. In particular, we show that when FIFO is an optimal issuing policy, moving units from older to younger positions in the inventory vector can only decrease the expected total cost. This result is very intuitive and helps significantly simplify the analysis by allowing properly modifying the inventory vectors and effectively matching units under two different policies. We believe these ideas are valuable beyond this study and can also be applied to facilitate the analysis for other perishable inventory problems.

CHAPTER 3

PERISHABLE INVENTORY SHARING IN A TWO-LOCATION SYSTEM

3.1 Introduction

As a form of risk pooling, inventory sharing has been well known to be an effective way for reducing costs and risks in multi-location inventory systems. Inventory sharing across different locations or channels is widely implemented in practice due to ever increasing service level requirements, integrated information technology systems, and improved cost-efficiency for transshipments [38]. Research on inventory sharing can be dated back to the 1950s [39] and has been continuously active in the past decade (e.g. [40, 41, 42, 43]). However, this stream of literature has primarily focused on nonperishable products that can stay in inventory indefinitely, and research on perishable inventory sharing has been limited (see §4.2 and [44] for literature review).

Products that will perish or become outdated after a short period of time are common in practice (e.g. many medical products such as blood products and pharmaceuticals and many food products such as produce, milk and fresh meat). Perishability imposes several challenges in managing inventory systems, and it is well established from both theory and practice perspectives that different inventory policies are needed for perishable products [12, 45]. While there are numerous studies on perishable inventory management problems, these studies have mostly focused on single-location systems, and few studies have considered a perishable inventory sharing problem in multi-location systems (see §4.2 for a detailed literature review).

In this paper, we fill in this gap by studying a joint ordering and transshipment decision problem for perishable products in a two-location system. Our analysis is motivated by a platelet (a blood product with three days of shelf-life) inventory management problem in a local two-location hospital system. Each of the two hospitals makes daily ordering decisions to replenish their platelet inventory. Further, the two hospitals belong to the same integrated healthcare system, and there are couriers that travel between the two hospitals on an hourly basis, which provides natural opportunities for transshipments of platelets between the two hospitals. During our interactions with the blood bank managing team of this hospital network, we learned that the platelet demand patterns in these two hospitals are significantly different. In particular, at one hospital, cardiac surgeries are the main reasons for platelet transfusions. Since both the number of surgeries per day and the number of platelet units needed per surgery are highly varying, the demand for platelets at this hospital is highly uncertain. In contrast, the other hospital mainly serves hematology patients, for whom the demand for platelets is much more predictable. The established findings in the nonperishable inventory literature suggest that the smaller the variances of demands, the smaller the value of inventory sharing, and when the demand variance at one location goes to zero, the value of inventory sharing also goes to zero irrespective of the demand variance at the other location [40, 41]. However, whether these established findings in the nonperishable inventory sharing literature carry over to the perishable setting remains an open question. Other interesting questions include: If the results from the nonperishable inventory literature do not generalize, then in what ways would the results differ? What are the characteristics of efficient inventory management policies in multi-location perishable inventory systems?

To study these questions, we consider a periodic-review, two-location perishable inventory system where products can be transshipped between the two locations at each period after demand realization, and inventory at each location is replenished using a base-stock policy, a widely considered ordering policy for perishable inventory management [e.g., 17, 18, 19, 46]. Under given base-stock levels, we first determine the direction of transshipment based on the inventory levels and realized demands at each location. We then derive upper and lower bounds on the optimal transshipment quantity. For a special case with two-period lifetime, we prove that the optimal cost function is $L^\natural$-convex, which implies that the optimal transshipment quantity is monotone in the inventory levels at each location. We further show via a counterexample that this property, however, is violated in the general case with larger lifetimes. Finally, we present an intuitive and easy-to-implement transshipment policy (which has a simpler structure than the optimal policy), and derive approximations of the expected cost functions based on which we develop base-stock levels for both locations. Using real-life data from the platelet inventory management problem discussed above, we show that our proposed policy is very competitive and significantly reduces the total costs compared with benchmark policies.

Furthermore, by comparing our findings with the existing results from the multi-location nonperishable and single-location perishable inventory literature, we establish the following results with important managerial implications. First, we show that under given order-up-to levels, the optimal transshipment quantity for the nonperishable case provides a non-tight lower bound on that for the perishable case,[1] which implies that transshipment should occur more often or with a larger quantity in perishable inventory systems than in nonperishable ones. Second, while it is well established in the single-location setting that the ordering decisions in the perishable case should be more conservative than that in the nonperishable case (due to the concern of outdating) [11, 12], we find that this result does not necessarily translate to the two-location setting: when inventory sharing is possible, it may be optimal to order strictly more in the perishable case than in the nonperishable case. Last but not least, we find that the value of inventory sharing for the perishable case is typically higher than that for the nonperishable case. In particular, and contrary to the established finding in the nonperishable inventory literature that value of inventory sharing vanishes as the demand variance at one location goes to zero [40, 41], we find that the value of inventory sharing for the perishable case can be strictly positive and substantial even when

[1] In the nonperishable case, it has been shown that under mild conditions on cost parameters, transshipment only occurs when there is shortage at one location and surplus inventory at the other location and the optimal transshipment quantity is the minimum of the shortage and the surplus [47, 48].

demand at one location becomes deterministic.

The rest of the paper is organized as follows. In §4.2, we present an overview of related literature. In §4.3, we present a model formulation of the two-location perishable inventory sharing problem. In §3.4, we derive structural properties of an optimal transshipment policy. Then, in §3.5, we present an intuitive transshipment policy and derive near-optimal base-stock levels. Finally, we present results from extensive numerical analysis in §4.7 and draw conclusions in §4.8.

3.2 Literature Review

Our work is related to the stream of research that studies nonperishable inventory sharing problems in periodic-review, multi-location systems (see [49, 50] for overviews). In this context, [39, 51] and [52] considered two-location problems with transshipments in a single-period case, while [53, 54, 55, 47] and [48] analyzed the structure of optimal transshipment policies in multiperiod cases. In particular, in a nonperishable system where transshipment takes place after demand realization, it has been shown that under mild conditions on cost parameters, the optimal transshipment quantity is simply the minimum of the shortage at one location and the surplus inventory at the other location [47, 48]. This result has been later extended to consider capacity constraints under which it may be optimal to reserve some inventory at each location and not satisfy the shortage at the other location [41]. A number of other extensions have also been studied in this literature. For example, [40] considered a two-location capacitated system with virtual transshipments, [56] and [57] studied network design problems with transshipments, [42] examined the structure of optimal transshipment policies for inventory systems with lost sales, [58] derived asymptotics and bounds for a system with a one-time stocking decision at the beginning of the selling season and multiple transshipment decisions throughout, and [59, 60] and [61], among many others, considered transshipment decisions in decentralized systems. We contribute to, and extend the literature of inventory sharing by considering joint ordering and transshipment decisions for perishable products and establishing findings that are significantly different from those in nonperishable settings.

Our work also relates to the research on periodic-review, fixed-lifetime perishable inventory management at a single location (see [8, 9] and [10] for overviews). The general multi-period lifetime perishable inventory management problem was first studied independently by [11] and [12]. However, the structure of an optimal policy for this problem is complicated, and finding optimal policies using standard dynamic programming is computationally intractable due to the well-known "curse of dimensionality". Therefore, later efforts have mainly focused on heuristic policies. In particular, the base-stock policy, although in general not optimal in perishable inventory systems, has been widely considered in the perishable inventory literature due to its simplicity, practicality and near optimal performance [e.g. 13, 14, 15, 16, 17, 18, 19, 46]. In addition, [22] and [23] examined constant order policies, [24] presented high-order approximations, [62] considered approximate dynamic programming algorithms, and [32, 63, 64] and [65] studied approximation algorithms with performance guarantees, and a number of other studies considered perishable inventory management in the context of blood products, and most commonly platelets [e.g., 35, 66, 36]. Our study contributes to and extends this literature by considering a perishable inventory sharing problem at a two-location system.

While there is a large body of literature on both multi-location nonperishable and single-location perishable inventory management, research on inventory sharing for perishable products is limited (see [10, 44] and [67] for literature review). A number of studies examined inventory allocation decisions in multi-location perishable inventory systems [e.g. 68, 69, 70, 71]. However, transshipment decisions are not explicitly considered in these studies. Instead, it is assumed that at each period products at all locations are collected to a central warehouse (for free) and then reallocated. Another stream of literature studied inventory sharing in specific contexts such as the management of blood products or slow-moving medical products [e.g. 72, 73, 74, 75, 67]. However, most of these studies either restricted their attention to certain classes of demand distributions (e.g. Poisson demand), or considered restricted types of transshipments (e.g., allow transshipments to occur only when a shortage is observed), or focused on "what if" type questions using simulation models. To the best of our knowledge, the only study that has analytically studied a general joint ordering and transshipment decision problem for perishable products is by [44], who presented an efficient algorithm for finding the optimal ordering and transshipment policies. However, [44] restricted attention to a single-period setting where the inventory system dynamics are not captured. Further, unlike our study, [44] considered proactive transshipment where transshipment decisions are made before demand is realized and hence transshipment is not used to meet shortages. Our paper extends this literature by studying a multi-period joint ordering and transshipment decision problem for perishable products and systematically comparing our findings with those in the nonperishable inventory literature.

Finally, our work also relates to the stream of studies that apply the theory of $L^\natural$-convexity to analyze structural properties of optimal inventory policies. In particular, the concept of $L^\natural$-convexity was first introduced to the inventory management literature by [76], and first applied to establish structural properties of optimal policies in inventory systems by [77]. Later, [78] extended the analysis to serial inventory systems, [79] investigated joint pricing and inventory control problems, [19] and [80] studied a coordinated inventory control and pricing problem and a clearance sales problem, respectively, for single-location perishable inventory systems, and [43] examined a two-location inventory system for nonperishable products. We contribute to this literature by considering $L^\natural$-convexity in a two-location perishable inventory system under base-stock ordering policies. In this setting, we show that $L^\natural$-convexity holds for a special case where the product lifetime is equal to two, which requires a different analysis from the nonperishable case due to the outdating process. Further, and perhaps more interestingly, we show via a counterexample that $L^\natural$-convexity could be violated for the general case with larger lifetimes.

3.3 Model Formulation

We now present our model for the two-location perishable inventory sharing problem, where each of the locations faces stochastic demand, and inventory is replenished according to a base-stock policy. We assume that products ordered arrive instantly with a zero lead time, and unmet demand is lost. Further, we allow transshipments between the two locations, and similar to many existing studies of nonperishable inventory sharing problems (e.g. [40, 41]), we assume that products can be transshipped from one location to the other in each period after demands are realized but before they are satisfied. Below we formally define the model components.

Notation: Consider a product lifetime of $K > 1$ periods and a planning horizon of T periods. Let $i = 1, 2$ denote the index of each location. We now introduce some notation for describing the system dynamics:

$x^i_{k,t}$: inventory level of products of age k at location i at the beginning of period t, $k = 1, ..., K-1$.

$\mathbf{x}^i_t$: inventory vector at location i at the beginning of period t.

d^i_t: demand at location i at period t. We assume that demand at each location is independent and identically distributed (i.i.d.). Note that we allow demand across the two locations to be correlated. Further, while the main analysis in this paper is conducted under the i.i.d. demand assumption for tractability, our derived policy can be easily extended to capture nonstationary demand, and we test the performance of our proposed policy under nonstationary demand in §3.6.2.

S^i: base-stock level at location i (for the nonstationary case we allow S^i to be time dependent).

$x^i_{0,t}$: order quantity of fresh products (i.e., of age zero) at location i at the beginning of period t; then $x^i_{0,t} = (S^i - \sum_{k=1}^{K-1} x^i_{k,t})^+$, where $(\cdot)^+$ denotes $\max\{\cdot, 0\}$.

u_t: transshipment quantity from location 1 to location 2 at period t after demands are realized but before they are satisfied (negative u_t implies transshipments from location 2 to location 1). We assume that products are only shipped out from one location if they are used to meet demand at the other location. Then, $-d_t^1 \leq u_t \leq d_t^2$.

System Dynamics: The sequence of events is as follows: 1) at the beginning of each period t, the $K-1$ dimensional inventory vector $\mathbf{x}_t^i$ at each location is observed, based on which $x_{0,t}^i = (S^i - \sum_{k=1}^{K-1} x_{k,t}^i)^+$ units of fresh products are ordered (and arrive with a zero lead time); 2) demand at each location d_t^i is realized; 3) transshipment decisions are made based on the inventory levels $x_{k,t}^i, k = 0, ..., K-1$ and realized demands d_i^t at each location, and products are transshipped from one location to the other with a zero lead time in a First-In-First-Out (FIFO) manner, i.e., older products are shipped out first; 4) after transshipment, products at each location are issued to satisfy demand in a FIFO manner, and unmet demand is lost; 5) at the end of each period, all products left in inventory age by one, and the products reaching age K are disposed from inventory.

Clearly, products of age $k-1$ at each location i will only be consumed after all of the units of age greater than or equal to k are consumed (either used to meet demand or transshipped out), and the surplus inventory (i.e., inventory left after satisfying demand) of age $k-1$ becomes of age k at the next period. Therefore, the inventory vectors at the two locations are updated as follows:

$$x_{k,t+1}^1 = \left(x_{k-1,t}^1 - \left(d_t^1 + u_t - \sum_{l=k}^{K-1} x_{l,t}^1\right)^+\right)^+, k = 1, ..., K-1; \tag{3.1}$$

$$x_{k,t+1}^2 = \left(x_{k-1,t}^2 - \left(d_t^2 - u_t - \sum_{l=k}^{K-1} x_{l,t}^2\right)^+\right)^+, k = 1, ..., K-1. \tag{3.2}$$

Costs and Optimality Equations: At each period, we consider a shortage penalty p^i for each unit of stock-out at location i, a holding cost h^i for each unit of surplus inventory at location i, an outdating cost w^i for each unit of outdate at location i, and a transshipment cost r^i for each unit of transshipment from location i to location $-i$, where we use $-i$ to denote the location other than i. Without loss of generality, we assume zero unit ordering cost at each location, because any problem with a positive unit ordering cost can be transformed to an equivalent problem with zero ordering cost (see appendix for details).

Similar to several studies on nonperishable inventory sharing (e.g. [47, 41]), we assume that $p^i \leq p^{-i} + r^{-i}, i = 1, 2$, which implies that the unit shortage penalty at one location is no larger than the unit shortage penalty at the other location plus a unit transshipment cost. That is, when there are shortages at both locations, the total cost is no larger if a product is used to satisfy demand at its own location. Similarly, we assume that $h^i \leq h^{-i} + r^i, i = 1, 2$, and $h^i + w^i \leq h^{-i} + w^{-i} + r^i, i = 1, 2$, which imply that the unit holding (and outdating) cost at one location is no larger than the unit holding (and outdating) cost at the other location plus a unit transshipment cost. Finally, we assume that $r^i < h^i + p^{-i}, i = 1, 2$, so that it is worthwhile to trigger transshipments to meet shortages.

For simplicity, we assume that the system starts from zero inventory, i.e., $x^i_{k,1} = 0, i = 1, 2; k = 1, ..., K-1$. Then, given the base-stock levels S^1 and S^2, we have $\sum_{k=1}^{K-1} x^i_{k,t} \leq S^i, i = 1, 2$ at any period t, and the one-period cost function is defined as follows:

$$L_t(\mathbf{x}^1_t, \mathbf{x}^2_t, u_t) := p^1(d^1_t + u_t - S^1)^+ + p^2(d^2_t - u_t - S^2)^+ + h^1(S^1 - d^1_t - u_t)^+ + h^2(S^2 - d^2_t + u_t)^+$$

$$+ w^1(x^1_{K-1,t} - d^1_t - u_t)^+ + w^2(x^2_{K-1,t} - d^2_t + u_t)^+ + r^1(u_t)^+ + r^2(-u_t)^+.$$

Let $C_t(\mathbf{x}^1_t, \mathbf{x}^2_t)$ be the optimal expected cost-to-go function at period t, and define $C_{T+1}(\mathbf{x}^1_{T+1}, \mathbf{x}^2_{T+1}) = 0, \forall \mathbf{x}^1_{T+1}, \mathbf{x}^2_{T+1}$. Then, the optimality equation of our problem is defined as:

$$C_t(\mathbf{x}^1_t, \mathbf{x}^2_t) = \mathrm{E}\left[\min_{-d^1_t \leq u_t \leq d^2_t} \left\{ L_t(\mathbf{x}^1_t, \mathbf{x}^2_t, u_t) + \beta C_{t+1}(\mathbf{x}^1_{t+1}, \mathbf{x}^2_{t+1}) \right\} \right], t = 1, ..., T,$$

where β is a discount factor, the inventory vectors $\mathbf{x}^1_{t+1}$ and $\mathbf{x}^2_{t+1}$ are updated according to Equations 3.1-3.2, and the base-stock levels S^1 and S^2 are selected so that the expected total discounted cost $C_1(\mathbf{x}^1_1, \mathbf{x}^2_1)$ is minimized.

3.4 Structural Properties of Optimal Transshipment Policy

In this section, we first characterize the direction of optimal transshipment in §3.4.1. Then, in §3.4.2, we show that the optimal transshipment quantity in a nonperishable inventory problem provides a lower bound for that in a perishable inventory problem. In §3.4.3, we further show that when the product lifetime is equal to two, the optimal cost-to-go function is $L^\natural$-convex and hence the optimal transshipment quantity is monotone in the inventory level of each location. These results enable us to generate insightful findings on efficient perishable inventory management strategies, and lead to the development of a competitive transshipment policy, as presented in the next section. The proofs of all analytical results are presented in the appendix.

3.4.1 Direction of Optimal Transshipment

We first characterize the direction of optimal transshipment, i.e., whether products should be transshipped from location 1 to location 2 or vice versa, based on the inventory levels and realized demands at each location.

We start with a preliminary result, Lemma 7. For any inventory vectors $\mathbf{x}_t^1$ and $\mathbf{x}_t^2$ at the two locations and for $k = 1, ..., K-1$, define $C_t^{(k)}(\mathbf{x}_t^1, \mathbf{x}_t^2) := \partial C_t(\mathbf{x}_t^1, \mathbf{x}_t^2)/\partial x_{k,t}^1$ and $\hat{C}_t^{(k)}(\mathbf{x}_t^1, \mathbf{x}_t^2) := \partial C_t(\mathbf{x}_t^1, \mathbf{x}_t^2)/\partial x_{k,t}^2$ for the continuous case, where we use $\partial y(x)/\partial x$ to denote the right partial derivative of $y(x)$ with respect to x throughout the paper; define $C_t^{(k)}(\mathbf{x}_t^1, \mathbf{x}_t^2) := C_t(\mathbf{x}_t^1 + \mathbf{e}_k, \mathbf{x}_t^2) - C_t(\mathbf{x}_t^1, \mathbf{x}_t^2)$ and $\hat{C}_t^{(k)}(\mathbf{x}_t^1, \mathbf{x}_t^2) := C_t(\mathbf{x}_t^1, \mathbf{x}_t^2 + \mathbf{e}_k) - C_t(\mathbf{x}_t^1, \mathbf{x}_t^2)$ for the discrete case, where $\mathbf{e}_i$ is a $(K-1)$-dimensional vector with the ith entry equal to 1 and all other entries equal to 0.

Lemma 7 *At any period t, for $1 \leq j < k \leq K-1$, we have $0 \leq C_t^{(j)}(\boldsymbol{x}_t^1, \boldsymbol{x}_t^2) \leq C_t^{(k)}(\boldsymbol{x}_t^1, \boldsymbol{x}_t^2) \leq w^1$, and $0 \leq \hat{C}_t^{(j)}(\boldsymbol{x}_t^1, \boldsymbol{x}_t^2) \leq \hat{C}_t^{(k)}(\boldsymbol{x}_t^1, \boldsymbol{x}_t^2) \leq w^2$, $\forall \boldsymbol{x}_t^1, \boldsymbol{x}_t^2$ such that $\sum_{k=1}^{K-1} x_{k,t}^1 < S^1, \sum_{k=1}^{K-1} x_{k,t}^2 < S^2$.*

Lemma 7 implies that an additional unit of inventory (of any age) at location i leads to an increase of the expected total cost that is bounded by w^i, and the older the additional unit of inventory, the larger the increase of the total cost. We remark that when $w^i = 0, i = 1, 2$, the problem reduces to the nonperishable case, which we discuss further in §3.4.2.

With this result, we now characterize the direction of optimal transshipment. At any period t, given the inventory levels $x^i_{k,t}, k = 1, ..., K-1$, and realized demands d^i_t at each location $i = 1, 2$, let u^*_t denote the optimal transshipment quantity.[2] The direction of optimal transshipment (i.e., the sign of u^*_t) is presented in the following proposition and summarized in Figure 3.1.

Proposition 4 *At any period t, when $d^1_t < S^1, d^2_t > S^2$ (i.e., regions 1,2 in Figure 3.1), then $u^*_t > 0$; when $d^1_t \leq x^1_{K-1,t}, x^2_{K-1,t} < d^2_t \leq S^2$ (i.e., region 4), then $u^*_t \geq 0$; when $d^1_t \leq x^1_{K-1,t}, d^2_t \leq x^2_{K-1,t}$, or $d^1_t \geq S^1, d^2_t \geq S^2$ (i.e., regions 3,7), then $u^*_t = 0$; when $x^1_{K-1,t} < d^1_t \leq S^1, d^2_t \leq x^2_{K-1,t}$ (i.e., region 8), then $u^*_t \leq 0$; when $d^1_t > S^1, d^2_t < S^2$ (i.e., regions 6,9), then $u^*_t < 0$; and when $x^1_{K-1,t} < d^1_t \leq S^1, x^2_{K-1,t} < d^2_t \leq S^2$ (i.e., region 5), u^*_t can be either positive, negative or zero.*

Proposition 4 is intuitive: for $i = 1, 2$, when demand at location i is smaller than its own base-stock level while demand at location $-i$ is larger than its own base-stock level (i.e., regions 1, 2 and 6, 9), products should be transshipped from location i to location $-i$ (i.e., $u^*_t > 0$ in regions 1, 2 and $u^*_t < 0$ in regions 6, 9). When demand at location i is smaller than the inventory level of the oldest products ($x^i_{K-1,t}$), while demand at location $-i$ is larger than the inventory level of the oldest products but smaller than its base-stock level (i.e., regions 4 and 8), products are either transshipped from location i to location $-i$ or no transshipment is needed (i.e., $u^*_t \geq 0$ for region 4 and $u^*_t \leq 0$ for region 8). When demands at the two locations are both larger than their own base-stock levels or both smaller than the inventory level of the oldest products (i.e., regions 3, 7), no transshipment is needed (i.e., $u^*_t = 0$). Finally, when demands at both locations are larger than the inventory level

[2]In the case of multiple optimal solutions, we break the tie by defining u^*_t as the optimal transshipment quantity with the smallest magnitude (in terms of absolute value).

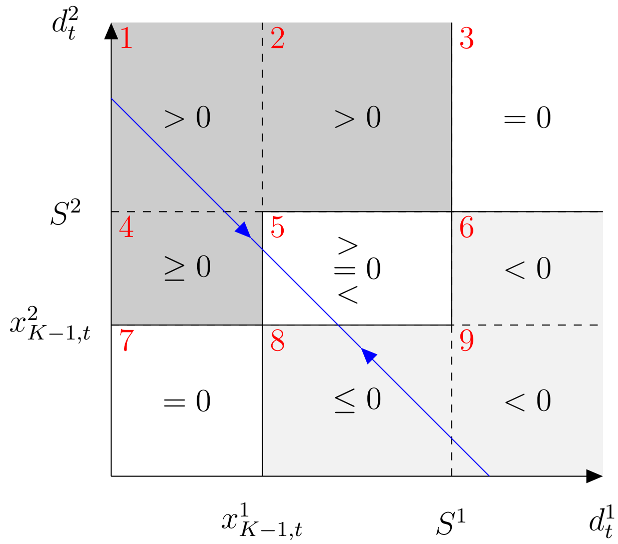


Figure 3.1: Direction of optimal transshipment for the perishable case (captured by the sign of u_t^*).

Note. Transshipping u_t units of products from location 1 to location 2 can be interpreted as transferring u_t units of demand from location 2 to location 1; with the transfer, the "artificial demands" at locations 1 and 2 are $d_t^1 + u_t$ and $d_t^2 - u_t$, respectively. Under this setup, a transshipment from location 1 (2) to location 2 (1) can simply be interpreted as moving the demands (d_t^1, d_t^2) on the -45 degree line in the southeast (northwest) direction.

of the oldest products but smaller than the base-stock level (i.e., region 5), transshipment can go in either direction depending on the inventory vectors, cost parameters and demand distributions.

Next, we further present an upper bound on the optimal transshipment quantity, which is closely related to the direction of optimal transshipment presented in Proposition 4.

Proposition 5 $-\min\{(d_t^1 - x^1_{K-1,t})^+, (S^2 - d_t^2)^+\} \leq u_t^* \leq \min\{(S^1 - d_t^1)^+, (d_t^2 - x^2_{K-1,t})^+\}$.

Proposition 5 can be described using Figure 3.1 if transshipment of products is interpreted as transfer of demands: transshipping u_t units of products from location 1 to location 2 is equivalent to transferring u_t units of demand from location 2 to location 1; with the transfer, demands at locations 1 and 2 become $d_t^1 + u_t$ and $d_t^2 - u_t$, respectively; call these quantities the "artificial demands". Under this setup, a transshipment from location 1 to location 2 can simply be interpreted as moving the demands (d_t^1, d_t^2) on the -45 degree line in the southeast direction in Figure 3.1, and vice versa. Then, Proposition 5 says that if the demands (d_t^1, d_t^2) lie in regions 1, 2, 4 or 5, (i.e., $d_t^1 \leq S^1$ and $d_t^2 \geq x_{K-1,t}^2$), then the optimal transshipment quantity u_t^* is such that the artificial demands $(d_t^1 + u_t^*, d_t^2 - u_t^*)$ must also lie in regions 1, 2, 4 or 5. Similarly, if (d_t^1, d_t^2) lie in regions 5, 6, 8 or 9, then $(d_t^1 + u_t^*, d_t^2 - u_t^*)$ must also lie in regions 5, 6, 8 or 9. The underlining intuition of this structure is that if the transshipment quantity goes beyond this upper bound, a location will be transshipping out products when there is shortage at its own location or when there is outdate at the other location, which is clearly not preferred.

3.4.2 Comparison with the Nonperishable Case: A Lower Bound on Optimal Transshipment Quantity

In this subsection, we first present an optimal transshipment quantity for a nonperishable inventory problem, and show that this quantity i) is optimal for the perishable case when the unit outdating costs are sufficiently small, and ii) provides a lower bound for the perishable case in general.

In the nonperishable inventory literature, it is well established under mild conditions on the cost parameters that transshipment occurs only when there is shortage at one location and surplus inventory at the other location, and the optimal transshipment quantity is equal to the minimum of the shortage and the surplus. In the following proposition, we extend this result to the perishable inventory case when the unit outdating costs $w^i, i = 1, 2$ are sufficiently small (recall that when $w^1 = w^2 = 0$, our problem becomes equivalent to a nonperishable inventory problem). In particular, let u_t^{N*} denote the optimal transshipment quantity for the nonperishable case, then we have the following result:

Proposition 6 *Suppose $w^i \le r^i - h^i + h^{-i}, i = 1, 2$. Then:*

$$u_t^* = u_t^{N*} = \begin{cases} \min\{(S^1 - d_t^1)^+, (d_t^2 - S^2)^+\}, \\ \quad \text{if } S^1 \ge d_t^1, d_t^2 \ge S^2; \\ -\min\{(d_t^1 - S^1)^+, (S^2 - d_t^2)^+\}, \\ \quad \text{if } d_t^1 \ge S^1, S^2 \ge d_t^2; \\ 0, \ \text{otherwise}. \end{cases} \tag{3.3}$$

Proposition 6 says that when the unit outdating costs are sufficiently small compared with the unit transshipment costs, the optimal transshipment policy for the perishable case is the same as that for the nonperishable case. The structure of such a policy is presented in Figure 3.2. In this case, if the demands (d_t^1, d_t^2) lie in regions 1, 2, then the artificial demands $(d_t^1 + u_t^{N*}, d_t^2 - u_t^{N*})$ must lie at the boundary between regions 1, 2 and regions 3, 4 or 5. Similarly, if (d_t^1, d_t^2) lie in regions 6, 9, then $(d_t^1 + u_t^{N*}, d_t^2 - u_t^{N*})$ must lie at the boundary between regions 6, 9 and regions 3, 5 or 8.

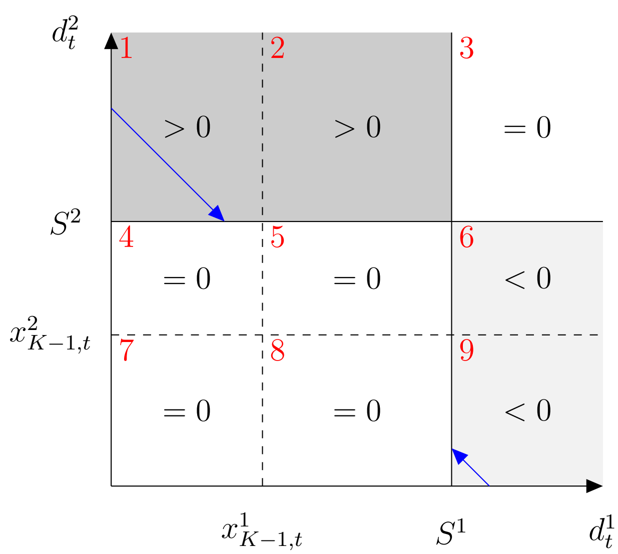


Figure 3.2: Direction of optimal transshipment for the nonperishable case or the perishable case with sufficiently small outdating costs (captured by the sign of u_t^{N*}).

When the unit outdating costs become large, however, results can be very different.

We show in the following proposition that for the general case, the optimal transshipment quantity for the nonperishable case (Equation 3.3) provides a lower bound on that for the perishable case. We further present an example to show that this lower bound is non-tight, i.e., the optimal transshipment quantity for the perishable case can be strictly higher than that for the nonperishable case.

Proposition 7 $|u_t^*| \geq |u_t^{N*}|$.

Proposition 7 is intuitive because while transshipment can be used to meet shortages in a nonperishable system, it can also be used to meet shortages in a perishable system (given that other conditions are the same). That is, whenever there are transshipments in a nonperishable system, there should also be transshipments in a perishable system. However, the reverse is not true. As we show in the example below, transshipment can be valuable for the perishable case even if there is no shortage at either location.

Example 1 *Consider an instance where $K = 2$ and $T = 1$. Consider base-stock levels $S^1 = S^2 = 1$, and assume that the initial inventory levels are $x_1^1 = 1$ and $x_1^2 = 0$, respectively. Then after ordering, the inventory vectors at the two locations are $(0, 1)$ and $(1, 0)$, respectively. Suppose the realized demands at the two locations are $d_1^1 = 0$ and $d_1^2 = 1$, respectively. Then, there will be no shortage at either location and an optimal transshipment policy for the nonperishable case will suggest no transshipment. However, if no transshipment occurs, then there will be a unit of outdate at location 1. In contrast, if we transship one unit of product from location 1 to location 2, the unit of outdate can be avoided and the total cost will be smaller than no transshipment if per unit transshipment cost is smaller than per unit outdating cost, which is typically the case in perishable inventory systems.*

The situation presented in Example 1 lies in region 4 of Figures 3.1 and 3.2. Comparing Figures 3.1 and 3.2, we observe that a similar situation could also occur in regions 5 and 8, where the optimal transshipment quantity for the nonperishable case is always zero, while that for the perishable case could be strictly positive or negative. Moreover, the optimal transshipment quantity for the perishable case in regions 1, 2, 6 and 9 could also be strictly larger than that for the nonperishable case (in terms of absolute value). For instance, if (d_t^1, d_t^2) lie in regions 1, 2, the artificial demands will only hit the boundary between regions 1, 2 and regions 3, 4 or 5 under the optimal transshipment policy in the nonperishable case. However, this is not true for the perishable case, where the artificial demands could fall into the interior of regions 4 and 5.

The implication of the above results is that when managing perishable inventory, one should expect transshipments to occur more often or in larger quantities compared with the nonperishable case. This is because in the perishable case, transshipments are valuable not only for reducing shortages but also for balancing the age of the products at different locations so as to reduce outdates. We remark that although intuitive, this result is important as it provides significant practical implications, and to the best of our knowledge, it has not been formally characterized or highlighted in the existing literature of inventory sharing/transshipment.

3.4.3 Monotonicity of Optimal Transshipment Quantity

In this subsection, we explore how the optimal transshipment quantity changes with respect to the inventory levels at each location.

In the nonperishable inventory literature, it has been shown that the optimal transshipment quantity from location 1 to location 2 is non-decreasing in the inventory level at location 1 and non-increasing in the inventory level at location 2 [40, 41, 43]. Intuitively, one would expect the same result to hold also for the perishable case. Indeed, we were able to show via $L^\natural$-convexity that the monotonicity result holds for a special case with a two-period lifetime. However, somewhat counterintuitively, we also find that monotonicity could be violated for the general lifetime case, which we show through an example at the end of this subsection.[3]

Let $\mathbb{F}$ be either the real space $\mathbb{R}$ or the integer space $\mathbb{Z}$. Following [81] and [82], we first define submodularity and $L^\natural$-convexity on $\mathbb{F}^n$ as follows (note that our analysis is applicable to both continuous and discrete quantities). Let $\vee$ and $\wedge$ be the componentwise maximum and minimum respectively, $\mathbb{F}_+$ and $\mathbb{F}_-$ be the set of nonnegative and nonpositive elements in $\mathbb{F}$ respectively, and **e** be the vector of 1's.

Definition 1 (Submodularity) *A real-valued function $g(\boldsymbol{x})$ defined on a lattice $X \subseteq \mathbb{F}^n$ (i.e., $\forall \boldsymbol{x}, \boldsymbol{x}' \in X, \boldsymbol{x} \vee \boldsymbol{x}' \in X$ and $\boldsymbol{x} \wedge \boldsymbol{x}' \in X$) is submodular if $\forall \boldsymbol{x}, \boldsymbol{x}' \in X$:*

$$g(\boldsymbol{x}) + g(\boldsymbol{x}') \geq g(\boldsymbol{x} \vee \boldsymbol{x}') + g(\boldsymbol{x} \wedge \boldsymbol{x}').$$

Definition 2 ($L^\natural$-convexity) *A real-valued function $g(\boldsymbol{x})$ defined on an $L^\natural$-convex set $X \subseteq \mathbb{F}^n$ (i.e., $\forall \boldsymbol{x}, \boldsymbol{x}' \in X, \forall \alpha \in \mathbb{F}_+, \boldsymbol{x} \vee (\boldsymbol{x}' - \alpha \boldsymbol{e}) \in X$ and $(\boldsymbol{x} + \alpha \boldsymbol{e}) \wedge \boldsymbol{x}' \in X$) is $L^\natural$-convex if function $\psi(\boldsymbol{x}, \xi) = g(\boldsymbol{x} - \xi \boldsymbol{e})$, $\xi \leq 0$, is submodular on $X \times \mathbb{F}_-$.*

When $K = 2$, let x_t^i denote the inventory levels of age one at location i, and let $C_t(x_t^1, x_t^2)$ denote the optimal cost-to-go function at period t. We now perform a state transformation, and show that the optimal cost-to-go function under transformed state variables is $L^\natural$-convex, which leads to the monotonicity result. In particular, let $s_t^1 := x_t^1$, $s_t^2 := -x_t^2$, and define $\tilde{C}_t(s_t^1, s_t^2) := C_t(s_t^1, -s_t^2), t = 1, ..., T$. Then:

Theorem 3 *When $K = 2$, the optimal cost-to-go function $\tilde{C}_t(s_t^1, s_t^2)$ is $L^\natural$-convex.*

In the following corollary, we show that the optimal transshipment quantity is monotone in the inventory levels at the two locations, and the sensitivity is bounded by one [77]. More specifically, when the inventory level at location 1 (2) increases by one, the optimal transshipment quantity would increase (decrease) and increases (decreases) at most by one.

[3] We also remark that our proof for $L^\natural$-convexity for the two-period lifetime case relies on the base-stock ordering policy assumption and does not extend to general ordering policies.

Corollary 3 *When $K = 2$, $u_t^*(x_t^1, x_t^2)$ is non-decreasing in x_t^1 and non-increasing in x_t^2, where $u_t^*(x_t^1 + \delta, x_t^2) \leq u_t^*(x_t^1, x_t^2) + \delta$, and $u_t^*(x_t^1 + \delta, x_t^2) \geq u_t^*(x_t^1, x_t^2) - \delta$, for $\delta > 0$.*

As discussed above, while the monotonicity result holds for the case with a two-period lifetime, it does not extend to the general lifetime case, which is shown through the following example:

Example 2 *Consider an instance where $K = 3$ and $T = 2$. Suppose demand is distributed as follows: $\mathrm{P}(d_t^i = 0) = \epsilon, \mathrm{P}(d_t^i = 1) = 1 - 2\epsilon, \mathrm{P}(d_t^i = 2) = \epsilon, i = 1, 2; t = 1, 2$, where ϵ is a small number. Consider base-stock levels $S^1 = S^2 = 2$. Then, there will be no shortage at both locations as the demands are no larger than 2. Assume that the initial inventory levels at the two locations are $x_{1,1}^1 = 0, x_{2,1}^1 = 1$, and $x_{1,1}^2 = 2, x_{2,1}^2 = 0$, respectively. Then after ordering, the inventory vectors at the two locations are $(1, 0, 1)$ and $(0, 2, 0)$, respectively. Consider outdating costs $w^1 = w^2 = 5$, transshipment costs $r^1 = r^2 = 1$, zero holding costs, and discount factor $\beta = 1$ (i.e., no discount). Suppose that the realized demands at $t = 1$ are $d_1^1 = 0$ and $d_1^2 = 1$. If the transshipment quantity $u_1 = 0$, then there will be one unit of outdate at location 1 and it is not difficult to check that the expected total cost in the system is $5 + o(\epsilon)$. If $u_1 = 1$, then there should be one unit of transshipment from location 2 to location 1 at $t = 2$ with probability at least $(1 - 2\epsilon)^2$ (otherwise, there will be one unit of outdate at location 2). In this case, the expected total cost is $2 + o(\epsilon)$. Therefore, the optimal transshipment quantity is $u_1^* = 1$. However, if the inventory level at location 1 increases to $x_{1,1}^1 = 1$, then after ordering, the inventory vectors at the two locations become $(0, 1, 1)$ and $(0, 2, 0)$, respectively. If $u_1 = 0$, then the expected total cost in the system is $5 + o(\epsilon)$. If $u_1 = 1$, then there will be one unit of outdate at location 2 at $t = 2$ with probability at least $(1 - 2\epsilon)^2$, and hence the expected total cost is $6 + o(\epsilon)$. Therefore, the optimal transshipment quantity becomes $u_1^* = 0$.*

Example 2 illustrates that when the inventory level at location 1 increases, the optimal transshipment quantity from location 1 to location 2 may decrease. The main reasoning is that when the total inventory level at location 1 is not too high, transshipping some units out may help reduce outdates and hence is valuable. However, when the inventory level at location 1 becomes high, a transshipment may only delay outdates instead of avoiding them while also incurring an additional transshipment cost. This result implies that the established monotonicity result for the nonperishable case does not extend to the perishable case in general, and that the optimal transshipment policy for the perishable case can be complicated. In the next section, we present a simple transshipment policy that satisfies the monotonicity property (hence not optimal in general), and we show via extensive numerical analyses that it has a very competitive performance.

3.5 A Simple Transshipment Policy and Approximations on One-Period Cost

In this section, we first present an intuitive transshipment policy in §3.5.1, and show that our proposed transshipment policy provides a lower bound on the optimal transshipment quantity when the unit outdating costs are sufficiently large. Then, in §3.5.2, we derive approximations on the expected costs, which are used to compute the base-stock levels at both locations.

3.5.1 A Simple Transshipment Policy for Reducing Both Shortages and Outdates

As shown in §3.4.2, transshipment in the perishable case can be used not only for reducing shortages but also for reducing outdates. We next present a simple intuitive transshipment policy, under which transshipment is triggered under either of the following two circumstances: i) without transshipment, there is shortage at one location while there is surplus inventory at the other location; and ii) without transshipment, there is outdate at one location while younger products are consumed at the other location. Specifically, given the inventory levels $x_{k,t}^i$ and realized demands d_t^i at each location, we denote our transshipment quantity at period t as $\hat{u}_t$, which is defined as follows:

$$\hat{u}_t := \begin{cases} \max\{\min\{(S^1 - d_t^1)^+, (d_t^2 - S^2)^+\}, \min\{(x_{K-1,t}^1 - d_t^1)^+, (d_t^2 - x_{K-1,t}^2)^+\}\}, \\ \quad \text{if } S^1 \geq d_t^1, d_t^2 \geq S^2, \text{or } x_{K-1,t}^1 \geq d_t^1, d_t^2 \geq x_{K-1,t}^2; \\ -\max\{\min\{(d_t^1 - S^1)^+, (S^2 - d_t^2)^+\}, \min\{(d_t^1 - x_{K-1,t}^1)^+, (x_{K-1,t}^2 - d_t^2)^+\}\}, \\ \quad \text{if } d_t^1 \geq S^1, S^2 \geq d_t^2, \text{or } d_t^1 \geq x_{K-1,t}^1, x_{K-1,t}^2 \geq d_t^2; \\ 0, \text{ otherwise.} \end{cases} \tag{3.4}$$

The structure of our policy $\hat{u}_t$ is presented in Figure 3.3. Under our policy, if the demands (d_t^1, d_t^2) lie in regions 1, 2, 4, then the artificial demands $(d_t^1 + \hat{u}_t, d_t^2 - \hat{u}_t)$ must lie at the boundary between regions 1, 2, 4 and regions 3, 5 or 7. Similarly, if (d_t^1, d_t^2) lie in regions 6, 8, 9, then $(d_t^1 + \hat{u}_t, d_t^2 - \hat{u}_t)$ must lie at the boundary between regions 6, 8, 9 and regions 3, 5 or 7. Clearly, $\hat{u}_t$ lies between the lower and upper bounds we derived in §3.4 on the optimal transshipment quantity.

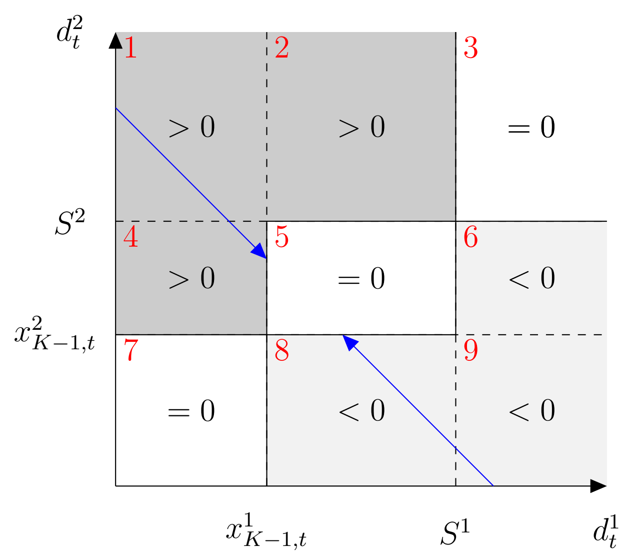


Figure 3.3: Direction of transshipment under our policy (captured by the sign of $\hat{u}_t$).

Comparing Figures 3.1 and 3.3, we observe that the main differences between our policy $\hat{u}_t$ and the optimal policy u_t^* lie in regions 4, 5, 8. First, in regions 4 and 8, $|\hat{u}_t|$ is always strictly positive, while u_t^* may be zero (e.g. when the unit outdating costs are sufficiently small as shown in Proposition 6). However, we later show that the gap between our policy and the optimal policy in these two regions disappears when the unit outdating cost is sufficiently large, as is typical in perishable inventory problems (see Proposition 8). Second, in region 5, $\hat{u}_t$ is always zero, while u_t^* may be strictly positive or negative. That is, younger products (i.e., products of age less than $K-1$) will never be transshipped under our policy when there is no shortage, while these products may be transshipped under an optimal policy. Clearly, the structure of our policy is simpler than that of an optimal policy, but we later show via extensive numerical analyses that such a simplification comes only with a small loss of optimality. This is mainly because transshipments of the oldest products are the most critical for reducing outdates, which have been captured by our policy. While transshipments of younger products can also be beneficial in balancing the inventory at the two locations, such transshipments can be largely made up by future transshipment opportunities.

Comparing Figures 3.2 and 3.3, we observe that the main difference between our policy $\hat{u}_t$ and the optimal policy for the nonperishable case u_t^{N*} lies in regions 4 and 8, where $|\hat{u}_t|$ is always strictly positive, while u_t^{N*} is always zero. In these two regions, the demand at each location is no larger than its own base-stock level, hence there is no shortage at either location. However, as the demand at one location is smaller while that at the other location is larger than the inventory level of the oldest products, there will be outdates at one location while younger products are consumed at the other location. The difference between our policy and u_t^{N*} in these two regions suggests that while transshipment is only triggered when a shortage is observed under u_t^{N*}, our policy also uses transshipment to redistribute the inventory of the oldest products to reduce outdates; we show in §4.7 that this difference leads to a substantial cost reduction.

Next, we show that under mild conditions, our policy provides a lower bound on the optimal transshipment quantity (clearly, a tighter lower bound than u_t^{N*}).

Proposition 8 *Suppose* $w^i > r^i - h^i + h^{-i} + \beta w^{-i}, i = 1, 2$. *Then* $|u_t^*| \geq |\hat{u}_t|$.

Proposition 8 implies that when the unit outdating costs are sufficiently large compared with the unit transshipment costs, our policy provides a lower bound on the optimal policy. In this case, the only difference between our policy and the optimal policy lies in region 5, where the optimal transshipment quantity can be either positive, zero or negative, while the transshipment quantity under our policy is always zero. That is, while younger products may also be transshipped for reducing (future) outdates under an optimal policy, we only transship the oldest products, which is intuitive and appealing from a practical perspective.

3.5.2 Approximations on the One-Period Cost

Given the transshipment quantity $\hat{u}_t$, we now derive approximations of the one-period cost function that depend on the inventory levels of the two locations only through the base-stock levels. These approximations are then used to determine the base-stock levels at both locations. As we show in §4.7, the ordering decisions in the two-location perishable setting can be significantly different from those in the two-location nonperishable and single-location perishable settings.

Given the base-stock levels S^1, S^2, demand realizations d_t^1, d_t^2 and transshipment quantity $\hat{u}_t$ (defined in Equation 3.4), the single-period cost is as follows:

$$L_t(\mathbf{x}_t^1, \mathbf{x}_t^2, \hat{u}_t) = p^1(d_t^1 + \hat{u}_t - S^1)^+ + p^2(d_t^2 - \hat{u}_t - S^2)^+ + h^1(S^1 - d_t^1 - \hat{u}_t)^+ + h^2(S^2 - d_t^2 + \hat{u}_t)^+$$

$$+ w^1(x_{K-1,t}^1 - d_t^1 - \hat{u}_t)^+ + w^2(x_{K-1,t}^2 - d_t^2 + \hat{u}_t)^+ + r^1(\hat{u}_t)^+ + r^2(-\hat{u}_t)^+.$$

Recall that under our transshipment policy, transshipment is used for reducing both shortages and outdates. Define $\hat{u}_t^S$ as the transshipment quantity if transshipment is only considered when a shortage is observed, and define $\hat{u}_t^O$ as the additional transshipment quantity that is used for reducing outdates. Then $\hat{u}_t^S = u_t^{N*}$ (defined in Equation 3.3), and $\hat{u}_t^O = \hat{u}_t - \hat{u}_t^S$.

To come up with approximations of $L_t(\mathbf{x}_t^1, \mathbf{x}_t^2, \hat{u}_t)$ that depend on the inventory levels of both locations only through the base-stock levels, we first observe that if the system starts with zero inventory, i.e., $x_{k,t}^i = 0, k = 1, ..., K-1; i = 1, 2$, then $\hat{u}_t = \hat{u}_t^S$, and the one-period cost is as follows:

$$\Gamma_t(S^1, S^2) := L_t(\mathbf{0}, \mathbf{0}, \hat{u}_t^S) = p^1(d_t^1 + \hat{u}_t^S - S^1)^+ + p^2(d_t^2 - \hat{u}_t^S - S^2)^+ + h^1(S^1 - d_t^1 - \hat{u}_t^S)^+$$

$$+h^2(S^2 - d_t^2 + \hat{u}_t^S)^+ + r^1(\hat{u}_t^S)^+ + r^2(-\hat{u}_t^S)^+.$$

Clearly, $\Gamma_t(S^1, S^2)$ depends on the inventory vectors only through the base-stock levels, and the one-period shortage penalty, holding cost and transshipment cost incurred for meeting shortages are well captured in $\Gamma_t(S^1, S^2)$. However, no outdating cost or transshipment cost incurred for reducing outdates is charged in $\Gamma_t(S^1, S^2)$. Since products ordered at period t will not outdate until period $t + K - 1$, we next define our first approximation of the one-period cost function by including the outdating costs and the transshipment costs incurred for reducing outdates at period $t + K - 1$ into the cost function at t [12, 13]. In particular, define:

$$O_{t+K-1}(S^1, S^2) := w^1(x_{K-1,t+K-1}^1 - d_{t+K-1}^1 - \hat{u}_{t+K-1}^O)^+ + w^2(x_{K-1,t}^2 - d_{t+K-1}^2 + \hat{u}_{t+K-1}^O)^+$$

$$+r^1(\hat{u}_{t+K-1}^O)^+ + r^2(-\hat{u}_{t+K-1}^O)^+,$$

where $x_{K-1,t+K-1}^i$ is the inventory level of age $K - 1$ at location i at period $t + K - 1$. Assuming that the system starts with zero inventory, there will be no outdates at any period $t, ..., t + K - 2$. Then, transshipments only occur to meet shortages at these periods, i.e., $\hat{u}_\tau = \hat{u}_\tau^S, \tau = t, ..., t + K - 2$. In this case, the inventory level of age $K - 1$ at location i at period $t + K - 1$ can be easily determined as:

$$x_{K-1,t+K-1}^i = \left(S^i - \sum_{\tau=t}^{t+K-2} \left(d_\tau^i + (d_\tau^{-i} - S^{-i})^+\right)\right)^+, i = 1, 2,$$

which clearly depends on the inventory levels of both locations only through the base-stock levels. Therefore, we define our first approximation of the one-period cost function as follows:

$$\Pi_1(S^1, S^2) := \Gamma_t(S^1, S^2) + \beta^{K-1} O_{t+K-1}(S^1, S^2).$$

As shown by [13], in a single-location system with no transshipment, the counterpart of $O_{t+K-1}(S^1, S^2)$, i.e., $w^i(x^i_{K-1,t+K-1} - d^i_{t+K-1})^+ = w^i(S^i - \sum_{\tau=t}^{t+K-1} d^i_\tau)^+$, overestimates the outdating cost at period $t+K-1$ with probability one (for any given initial inventory at period t and demand realizations at periods $t, ..., t+K-1$). This is because when the system starts with zero inventory, products will be consumed at a slower rate than otherwise, and hence there will be more units of age $K-1$ left in inventory at the beginning of period $t+K-1$, leading to a larger amount of outdates at $t+K-1$.[4] Based on this observation, we expect that $\mathrm{E}[\Pi_1(S^1, S^2)]$ tends to overestimate the expected one-period cost.

To address this potential overestimation, we next present a second approximation of the one-period cost which approximates the one-period outdating cost as the average of the total outdating costs across all periods $t, ..., t+K-1$ (instead of the outdating cost at $t+K-1$) [15]. In particular, if the system starts with zero inventory at period t, then there will be no outdates at periods $t, ...t+K-2$, and all outdates in periods $t, ..., t+K-1$ will occur at $t+K-1$. Therefore, we define our second approximation of the one-period cost function as follows:

$$\Pi_2(S^1, S^2) := \Gamma_t(S^1, S^2) + \alpha\beta^{K-1} O_{t+K-1}(S^1, S^2).$$

where $\alpha = \frac{1-\beta}{1-\beta^K}$ if $\beta < 1$, and $\alpha = \frac{1}{K}$ if $\beta = 1$.

As shown by [15], in a single-location system with no transshipment, the counterpart of $\alpha\mathrm{E}[O_{t+K-1}(S^1, S^2)]$ underestimates the long-run average one-period outdating cost. This

[4] We remark that in the two-location setting however, it is not always guaranteed that $O_{t+K-1}(S^1, S^2)$ overestimates the sum of the outdating costs and the transshipment costs incurred for reducing outdates at period $t+K-1$.

is because by assuming that the system starts with zero inventory t, the total amount of outdates at periods $t, ..., t+K-1$ will be smaller than otherwise. Based on this observation, we expect that $\mathrm{E}[\Pi_2(S^1, S^2)]$ tends to underestimate the expected one-period cost.

Given that $\mathrm{E}[\Pi_1(S^1, S^2)]$ tends to overestimate while $\mathrm{E}[\Pi_2(S^1, S^2)]$ tends to underestimate the expected one-period cost, we take the average of these two approximations and determine the base-stock levels of both locations by minimizing the average cost. In particular, let:

$$\bar{\Pi}(S^1, S^2) := \frac{\Pi_1(S^1, S^2) + \Pi_2(S^1, S^2)}{2}.$$

Then, the base-stock levels can be determined as $(\hat{S}^1, \hat{S}^2) \in \arg\min_{(S^1, S^2)} \mathrm{E}[\bar{\Pi}(S^1, S^2)]$.

Finally, we remark that although our analyses are conducted based on stationary demands, our transshipment and ordering policies derived in this section are both applicable to consider nonstationary demands, where we compute different base-stock levels for different periods.

3.6 Numerical Results

In this section, we numerically test the performance of our proposed policy and compare it with other benchmark policies. In particular, we first consider stationary demand in §3.6.1, and then consider nonstationary demand based on a platelet inventory management problem in §3.6.2.

3.6.1 Stationary Demand

In §3.6.1, we first consider a symmetric case where demands at the two locations are identically distributed. Then in §3.6.1, we consider an asymmetric case where demand at one location is variable while that at the other location is constant to mimic the motivating example in §3.1.

Symmetric Case

Similar to many existing studies on perishable inventory systems [e.g. 13, 16, 17], we consider short product lifetimes $K = 2, 3$ (while the optimal policy becomes computationally intractable for larger lifetimes, as we show in the appendix, our policy continues to perform better than other benchmark policies as lifetime gets larger). For each location, we consider two demand distributions, Poisson and geometric, to represent demand with small and large variances, respectively. We assume that the mean of demand is equal to 5, the discount factor is $\beta = 0.95$, and similar to [13], we consider the following cost parameters: $p = 5, 10$, $w = 5, 10$, $h = 0.5, 1$, and $r = 1, 2$. Considering all combinations of model parameters, we have a total of 64 problem instances. For each instance, we generate $10,000$ random scenarios and simulate a planning horizon of 100 periods.

We first benchmark our policy with an optimal policy, where the optimal transshipment policy is solved using standard application of dynamic programming and the optimal base-stock levels are found by simulation optimization. Let $C(\pi)$ and $C(\text{OPT})$ be the expected total discounted cost of a policy π and an optimal policy, respectively. We define the performance error of policy π as:

$$error(\pi) := \frac{C(\pi) - C(\text{OPT})}{C(\text{OPT})} \times 100\%.$$

The average performance error of our proposed policy across all 96 problem instances is 0.2% and the maximum performance error is 1.9%, which demonstrate that our policy has a very competitive performance. In order to quantify the value of inventory sharing and better understand the performance of our policy, we further benchmark our policy with the following three policies:

- No Sharing (NS): No transshipment is allowed, and each location follows a single-location optimal base-stock policy.

- Sharing Policy 1 (S1): Transshipment is conducted based on transshipment policy u_t^{N*} as defined in Equation 3.3, and each location follows a single-location optimal base-stock policy assuming no transshipment.

- Sharing Policy 2 (S2): Transshipment is conducted based on transshipment policy u_t^{N*} as defined in Equation 3.3, and base-stock levels are jointly determined at the two locations in a similar way as described in §3.5.2 (by plugging in u_t^{N*} instead of $\hat{u}_t$).

In particular, we benchmark with policy NS to assess the value of inventory sharing. We also benchmark with policies S1 and S2 because the improvement of S1 over NS captures the value of transshipment used for meeting shortages, and the improvement of S2 over S1 captures the value of jointly determining the base-stock levels at the two locations. Finally, the improvement of our policy over S2 captures the value of transshipment used for reducing outdates (note that in the nonperishable case, our policy and policy S2 are essentially the same and are optimal).

The expected total discounted cost of each policy under different model parameters are presented in Table 3.1. From these results, we first observe that all policies that allow transshipment perform significantly better than policy NS, which indicates that inventory sharing has a substantial value. Second, we observe that the marginal improvement of policy S2 over S1 is minor under most problem instances.[5] That is, under symmetric demand, the benefit of jointly determining ordering quantities is small under most circumstances (however as we show in §3.6.1 that this is not the case for asymmetric demands). Finally, and perhaps most interestingly, we observe a significant performance improvement from our policy over policy S2, an optimal policy for the nonperishable case. This result highlights the value of transshipment not only for meeting shortages but also for reducing outdates in the perishable case (and the value is especially large when the holding cost is

[5]Note that S2 may not always perform better than S1. This is because while the base-stock levels under S1 are optimal for single-location systems, the joint ordering decisions under S2 are made in a heuristic manner.

Table 3.1: Expected total costs under different policies: symmetric case (percentages of cost reduction over policy NS are included in parenthesis).

Policy			NS	S1	S2	Our Policy
Poisson	$K=2$	$p=5, h=0.5$	125.2	91.0 (27.3%)	87.5 (30.1%)	77.8 (37.9%)
		$p=5, h=1$	169.1	118.4 (30.0%)	118.4 (30.0%)	111.0 (34.4%)
		$p=10, h=0.5$	163.4	127.1 (22.2%)	105.5 (35.5%)	92.2 (43.6%)
		$p=10, h=1$	216.0	147.2 (31.9%)	147.8 (31.6%)	133.9 (38.0%)
	$K=3$	$p=5, h=0.5$	86.0	70.4 (18.2%)	63.7 (26.0%)	62.8 (27.0%)
		$p=5, h=1$	137.4	105.4 (23.3%)	102.5 (25.4%)	102.0 (25.8%)
		$p=10, h=0.5$	101.9	90.2 (11.5%)	72.3 (29.0%)	70.7 (30.6%)
		$p=10, h=1$	167.9	133.9 (20.3%)	119.8 (28.7%)	118.9 (29.2%)
Geometric	$K=2$	$p=5, h=0.5$	493.1	348.3 (29.4%)	345.8 (29.9%)	329.6 (33.2%)
		$p=5, h=1$	546.2	390.3 (28.5%)	390.3 (28.5%)	376.6 (31.1%)
		$p=10, h=0.5$	716.9	477.0 (33.5%)	469.9 (34.5%)	446.6 (37.7%)
		$p=10, h=1$	804.8	541.8 (32.7%)	540.6 (32.8%)	519.7 (35.4%)
	$K=3$	$p=5, h=0.5$	365.8	245.0 (33.0%)	245.0 (33.0%)	231.1 (36.8%)
		$p=5, h=1$	443.8	310.9 (30.0%)	310.9 (30.0%)	300.0 (32.4%)
		$p=10, h=0.5$	507.5	326.0 (35.8%)	313.1 (38.3%)	292.6 (42.3%)
		$p=10, h=1$	629.2	418.3 (33.5%)	409.1 (35.0%)	388.5 (38.3%)

Note. Other cost parameters are $w=5, r=1$; the patterns for other outdating and transshipment costs are similar.

small, i.e., when the outdating cost constitutes the majority of the overage cost).

To examine the effect of perishability, we present the value of inventory sharing (i.e., the percentage of cost reduction of our policy over policy NS) under different unit transshipment costs for both perishable and nonperishable cases in Figure 3.4 (recall that in the nonperishable case, our policy is the same as policy S2 and is optimal). We observe that under both Poisson and geometric demand, the value of inventory sharing is significantly higher in the perishable case, which underlines the importance of transshipments for perishable products. Also note that when demand variance is small (i.e., Poisson demand), the value of inventory sharing decreases rapidly when the product lifetime increases. However, this is not the case when the demand variance is large (i.e., geometric demand), where the value of inventory sharing is even higher when $K=3$ than when $K=2$.

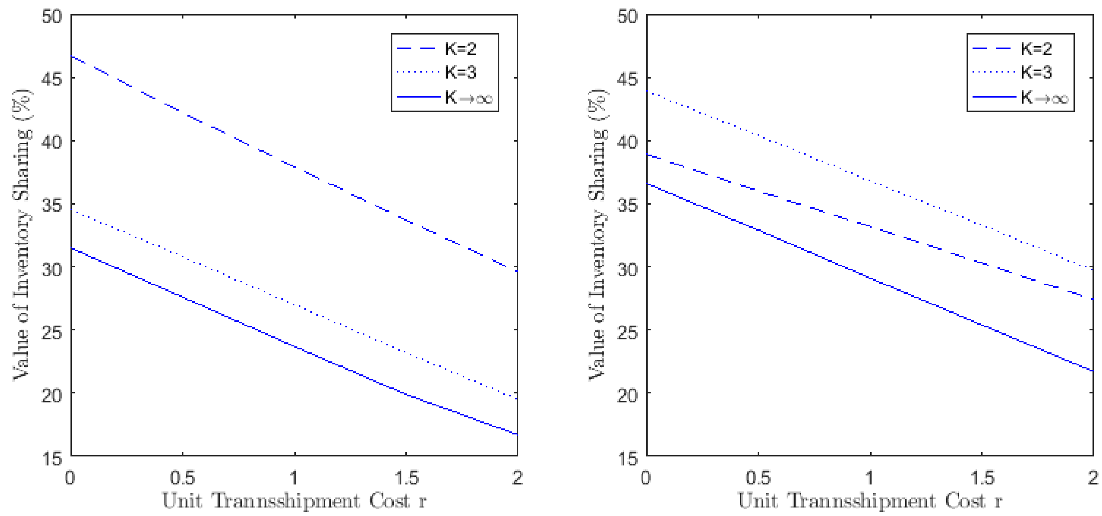


Figure 3.4: Value of inventory sharing under different unit transshipment costs for Poisson (left) and Geometric (right) demands. (The nonperishable case is captured by $K \to \infty$.)
Note. Other cost parameters are $p = 5, w = 5, h = 0.5$; the patterns for other costs are similar.

Asymmetric Case

To mimic the motivating example described in §3.1, we now consider an asymmetric case where demand at one location is variable while that at the other is constant. In particular, we consider two demand distributions for location 1, Poisson and geometric, both with mean equal to 5, and assume that demand at location 2 is deterministic and is equal to 5. Other parameters considered are the same as in the previous subsection. In this case, the average performance error of our proposed policy across all problem instances is 0.5% and the maximum performance error is 3.8%, which again illustrates the near-optimal performance of our policy.

The expected total discounted costs of different policies are presented in Table 3.2. From these results, we first observe that policy S1 always has the same expected total cost as policy NS. This is because under policy S1, the base-stock level at location 2 is always equal to 5, where there will be no shortage or surplus inventory after demand realization. Therefore, transshipment never occurs under policy S1 and the system dynamics of policies S1 and NS are exactly the same.

Table 3.2: Expected total costs under different policies: asymmetric case (percentages of cost reduction over policy NS are included in parenthesis).

Policy			NS	S1	S2	Our Policy
Poisson	$K=2$	$p=5, h=0.5$	62.6	62.6 (0.0%)	56.6 (9.5%)	47.3 (24.4%)
		$p=5, h=1$	84.4	84.4 (0.0%)	80.9 (4.1%)	71.3 (15.4%)
		$p=10, h=0.5$	82.0	82.0 (0.0%)	63.8 (22.2%)	55.1 (32.8%)
		$p=10, h=1$	107.9	107.9 (0.0%)	97.6 (9.5%)	88.3 (18.1%)
	$K=3$	$p=5, h=0.5$	43.1	43.1 (0.0%)	43.1 (0.0%)	41.4 (3.9%)
		$p=5, h=1$	68.7	68.7 (0.0%)	68.7 (0.0%)	68.0 (1.0%)
		$p=10, h=0.5$	51.2	51.2 (0.0%)	50.3 (1.8%)	47.7 (6.8%)
		$p=10, h=1$	84.1	84.1 (0.0%)	84.1 (0.0%)	82.4 (2.0%)
Geometric	$K=2$	$p=5, h=0.5$	246.6	246.6 (0.0%)	195.6 (20.7%)	181.4 (26.4%)
		$p=5, h=1$	273.1	273.1 (0.0%)	248.8 (8.9%)	218.6 (19.9%)
		$p=10, h=0.5$	358.3	358.3 (0.0%)	263.6 (26.4%)	260.8 (27.2%)
		$p=10, h=1$	402.3	402.3 (0.0%)	321.2 (20.2%)	317.5 (21.1%)
	$K=3$	$p=5, h=0.5$	183.2	183.2 (0.0%)	160.2 (12.5%)	139.6 (23.8%)
		$p=5, h=1$	222.1	222.1 (0.0%)	215.5 (3.0%)	194.8 (12.3%)
		$p=10, h=0.5$	254.0	254.0 (0.0%)	188.2 (25.9%)	173.0 (31.9%)
		$p=10, h=1$	315.0	315.0 (0.0%)	278.3 (11.7%)	258.2 (18.0%)

Note. Other cost parameters are $w=5, r=1$.

Second, somewhat surprisingly, we find that policy S2 exhibits a substantial performance improvement over S1. Given that S1 and S2 follow the same transshipment policy, the performance improvement must come from different base-stock levels. From Table 3.3, we observe that under policy NS or S1, the base-stock level at location 2 is always 5, i.e., equal to demand. However, under policy S2 or our policy, the base-stock level at location 2 can be strictly higher than 5. This may appear to be counterintuitive. However, there is a reasonable explanation: by ordering slightly more than 5 (but less than or equal to 10 for the case where $K=2$) at location 2, there will be no outdate at location 2. In this case, the surplus inventory at location 2 can be used as a buffer to meet the unmet demand at location 1 without incurring an outdating risk at location 2. Thus, the base-stock level at location 1 can be smaller than before and hence the amount of outdates at location 1 is reduced. This result highlights the importance of jointly determining the ordering quantities at different locations when demand is asymmetric. It also implies that the well-established result in the single-location setting that the optimal order-up-to level for the perishable case is always less than or equal to that for the nonperishable case [11, 12] does not translate to the two-location setting where transshipment is allowed.

Table 3.3: Base-stock levels under different policies for the asymmetric case where demand at location 2 is deterministic and equal to 5.

Policy			NS	S1	S2	Our Policy
Poisson	$K = 2$	$p = 5, h = 0.5$	(7,5)	(7,5)	(6,7)	(7,6)
		$p = 5, h = 1$	(6,5)	(6,5)	(6,6)	(7,5)
		$p = 10, h = 0.5$	(8,5)	(8,5)	(6,8)	(7,7)
		$p = 10, h = 1$	(7,5)	(7,5)	(6,7)	(7,6)

Third, similar to the symmetric case, we observe a further performance improvement of our policy over policy S2. More importantly, unlike policy S2 under which the value of inventory sharing relies on ordering strictly more than 5 units at location 2, there is a significant value of inventory sharing under our policy even if the ordering quantity at location 2 is exactly 5. This is because under our policy, transshipment is also used for reducing outdates. Then, even if there is neither shortage nor surplus inventory at location 2, location 2 can still help consume the old products at location 1 to reduce the amount of outdates while saving its own fresh products for future use.

Finally, from Figure 3.5, we observe that for the asymmetric case where demand at one location is deterministic, the value of inventory sharing is always zero for the nonperishable case, while it is strictly positive and substantial for the perishable case, which again shows that the established findings from the nonperishable inventory literature may not hold for the perishable case.

3.6.2 Nonstationary Demand: Platelet Inventory Management Problem in a Two-Hospital System

In this section, we test the performance of our proposed policy using real data from the platelet inventory management problem described in §3.1. In particular, at these two hospitals, 1) platelets are ordered on a daily basis and an order placed at the end of the previous

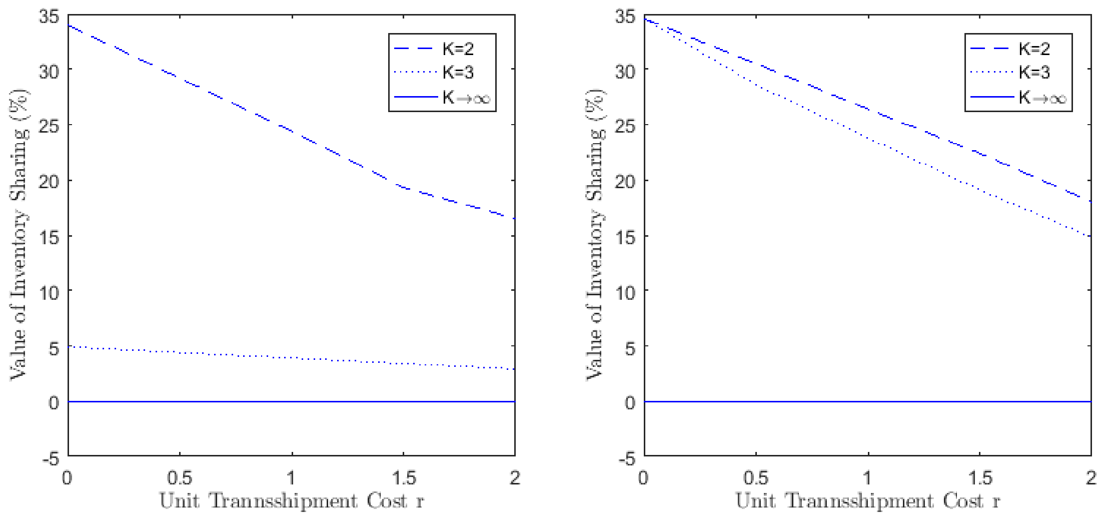


Figure 3.5: Value of inventory sharing under different unit transshipment costs for Poisson (left) and Geometric (right) demands. (The nonperishable case is captured by $K \to \infty$.)
Note. Other cost parameters are $p = 5, w = 5, h = 0.5$.

day arrives in the morning of the next day; 2) as demand arises, older products are typically issued first to reduce outdates; and 3) unmet demand is satisfied by emergency deliveries. Therefore, our assumptions for zero lead time, FIFO issuing policy, and lost sales are applicable in this setting.

Platelets have a short lifetime of $K = 3$ days. We consider a planning horizon of 4 weeks (i.e., $T = 28$ days). Similar to two recent studies on blood inventory management [35, 36], we assume that at each location, demand over time is independent but that the distribution in different days may not be identical. Based on the platelet transfusion data from October 2015 to September 2016 at these two hospitals, we estimate the daily demand at hospital 1 as a discretized normal distribution, and that at hospital 2 as a negative binomial distribution. The means and standard deviations of daily demand at the two hospitals are presented in Table 3.4. Further, we assume that demands at the two locations are independent, which is reasonable as these two hospitals serve different types of patients, as discussed in §3.1.

Using real data and in consultation with the blood bank managing team at this hospital

Table 3.4: Means and standard deviations for daily demand at the two hospitals.

		Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Hospital 1	Mean	26.1	24.7	23.6	24.0	25.2	17.4	17.4
	Stand. dev.	7.1	4.9	5.7	6.6	6.5	4.9	5.3
Hospital 2	Mean	4.6	6.3	5.5	5.7	6.3	3.1	3.1
	Stand. dev.	3.1	4.7	4.1	3.8	3.7	2.5	2.7

system, we estimated the unit outdating cost w to be equal to the purchase cost of $\$500$, unit holding cost h as $\$5$, unit transshipment cost r as $\$50$, and unit shortage penalty p as $\$2000$.[6] Similar as before, we benchmark our policy with policies NS, S1 and S2 (the optimal policy becomes computationally intractable in this case). The summary statistics and expected total costs of different policies are presented in Table 3.5. First, we observe that policy S1, which allows transshipment only when there is shortage, substantially reduces the expected amount of shortages compared with policy NS, but has negligible effect on the amount of outdates. Policy S2, which has the same transshipment policy as S1 but allows jointly determining base-stock levels at the two locations, substantially reduces the expected amount of outdates compared with policy S1 (with only a slight increase in the amount of shortages). Finally, our policy, under which transshipment is also triggered for reducing outdates, further significantly reduces the amount of outdates. Overall, the expected total cost reduction of our policy is substantial compared with policies NS (77.6%), S1 (62.5%) and S2 (26.9%).

3.7 Conclusion

In this paper, we study a joint ordering and transshipment decision problem in a two-location perishable inventory system. Our model and analysis provide the following three insights for the management of perishable inventory systems. First, we show that the opti-

[6] The shortage penalty for blood inventory management problems typically includes the cost of emergency shipments or the penalty of postponing blood transfusions, which is usually high and often estimated as 2-10 times higher than the purchase cost [35].

Table 3.5: Summary statistics and expected total costs within 4 weeks under different policies for the platelet inventory management problem (percentages of cost reduction over policy NS are included in parenthesis).

Policy	NS	S1	S2	Our Policy
# Shortage	4.18	0.13	0.28	0.27
# Outdate	16.26	16.20	3.43	0.38
# Transshipment	0	4.13	18.52	16.46
Total Cost ($)	19684	11729 (40.4%)	6024 (69.4%)	4400 (77.6%)

mal transshipment quantity for the perishable case is at least as high as (and can be strictly higher than) that for the nonperishable case. In particular, in the perishable case, even when there is no shortage, transshipment may be useful for balancing the inventory of old products to reduce the total amount of outdates. Second, in contrast to the single-location setting where the optimal order-up-to level for the perishable case is always smaller than or equal to that for the nonperishable case, we find that when inventory sharing is considered in a multi-location system, it may be optimal to order strictly more in the perishable case than in the nonperishable case. In particular, when demands at the two locations are asymmetric, the low variance location should increase the order quantity, which serves as a buffer for the high variance location, so that the amount of outdates at the high variance location can be reduced. Finally, we find that the value of inventory sharing for the perishable case is typically higher than for the nonperishable case. Interestingly, unlike in the nonperishable case, the value of inventory sharing in the perishable case can be strictly positive and substantial, even when demand at one location is deterministic. This is because in the perishable case, a deterministic location can help consume old products at the other location so as to reduce outdates. The implication of this result is that when products are perishable, transshipment should be considered even when existing results from the nonperishable inventory literature may suggest little or no value of transshipment.

Meanwhile, we present a simple and near-optimal inventory management policy for perishable inventory sharing that has the above characteristics and easily could be implemented in practice. Under our policy, transshipment is triggered when there is either shortage or immediate outdate. In this case, only the oldest products at each location will be transshipped unless there is shortage at the other location. Such a policy is very intuitive and considered practical by the blood inventory managers at the hospital system from which the study was inspired. As of writing this manuscript, we are also working closely with these managers to further develop decision support tools to help manage their platelet inventory. We believe that our policies and insights are also valuable in many other perishable inventory systems where inventory can be shared among multiple locations.

Our analysis can be extended in several directions. First, we assumed zero fixed ordering cost in our model. While this is reasonable for blood inventory management problems where orders are usually placed on a daily basis, there could be other settings where orders are not necessarily placed at every period and there is a fixed cost each time an order is placed. In that case, one needs to consider developing other ordering policies such as the (s, S) policy which has been widely used in inventory management problems with fixed ordering costs. Second, we did not consider capacity constraints in our analysis. In the nonperishable inventory literature, it has been shown that with capacity constraints on ordering decisions, it may be optimal to keep some safety stock at each location and not satisfy the shortage of the other location. We expect that similar results could be established for the perishable case; however, we do not anticipate such a policy to be very practical in perishable (especially blood) inventory management problems where the shortage penalty is typically very high. Third, we assumed FIFO issuing policy in this study. Our proposed transshipment policy, under which only the oldest products are transshipped, works very well in a FIFO model (which is reasonable when managing blood inventory). However, in other inventory systems where products are issued in different manners, e.g. Last-In-First-Out (LIFO), we anticipate that it may be important to also take younger products into consideration when transshipment decisions are made, and hence new inventory policies are needed.

CHAPTER 4
TRUTHFUL MECHANISMS FOR MEDICAL SURPLUS PRODUCT ALLOCATION

4.1 Introduction

In the United States, healthcare organizations annually dispose of 5.9 million tons of medical surplus products [83]. These products include many unused, unexpired medical supplies (e.g., leftovers from post-surgical procedures, unopened clinical kits, etc.) and used biomedical equipment (e.g., vital sign monitors, ultrasound units, infant incubators, etc.) that are typically discarded due to safety guidelines and regulatory requirements. Meanwhile, large portions of populations in the developing world suffer from inadequate medical supply and care. For example, the World Health Organization (WHO) estimates that close to 6 million children under the age of five died in 2015, mainly due to inadequate medical care [84]. Given this disparity between the developed and developing world with respect to the supply of medical products, there is a unique opportunity for bridging surplus with need to alleviate suffering in the developing world. In this context, Medical Surplus Recovery Organizations (MSROs) play a critical role by collecting and recovering medical surplus products in developed countries and redistributing them to healthcare organizations in medically under-served communities in developing countries.

While the supply of medical surplus products is limited in proportion to the high demand in the developing world, a significant portion of these products are ultimately wasted due to the mismatch between supply and demand. Indeed, the WHO estimates that for many recipients in low-resource countries, over seventy percent of donated medical equipment was inappropriate [85]. Delivering the right product to the right recipient in MSRO supply chains is particularly challenging for many reasons. First, unlike in a traditional for-profit supply chain where the supply can be managed to some extent through production, the supply of medical surplus heavily relies on donations, which are often uncertain and uncontrollable. Second, recipient needs for many critical medical products such as biomedical equipment are significant and largely heterogeneous. For example, a specific medical equipment may be critically needed by one recipient but not at all by another because of their respective medical specializations or operating conditions. In such situations, an inappropriate allocation of products would lead to a waste of critical resources. Third, MSROs usually serve large bases of recipients from a variety of different countries, and hence the information about the exact needs of recipients is typically very limited. Fourth, transportation is a major cost component for MSROs, and hence products are usually sent to recipients using full container shipments to achieve economies of scale, which may exacerbate the supply-demand mismatch. Last but not least, medical surplus products are allocated to recipients without using monetary transfers for several reasons (e.g., MSROs are nonprofit organizations with the objective to maximize value provision to recipients, in which case the use of financial mechanisms such as asking a recipient with a greater need to pay more is usually not considered a viable option [86]; also, recipients in this context are highly cash-constrained, which further precludes the use of monetary transfers).

Resource allocation problems without monetary transfer exist in other nonprofit settings such as school assignment and organ allocation as well. While the MSRO's resource allocation problem shares certain characteristics with these settings, the combination of challenges faced by MSRO supply chains differentiates an MSRO's problem from existing problems. For example, a school assignment problem typically considers a static setting where the number of slots for each school is given and fixed [87]. However, in the medical surplus setting, products are donated dynamically over time and in uncertain quantity. In the context of organ allocation, most existing studies either assume known patient types (which is reasonable because physicians assess their patient's condition prior to listing them on a transplant waitlist [88, 89]) or focus on the trade-off between a higher organ quality (which is recipient-independent) and a shorter waiting time [90]. In the medical surplus setting however, recipients' product preferences are typically unknown to the MSRO and largely heterogeneous. Moreover, in both school and organ allocation settings, each recipient receives only one item, while in the MSRO setting, recipients receive full containers of different product mixes. Due to these unique challenges, matching supply with demand is particularly difficult in an MSRO supply chain, leading to a significant wastage of the donated medical products [85].

In this context, a Southern U.S.-based MSRO, referred to as Beta, has developed an award-winning recipient-driven resource allocation model to better match medical surplus with recipient needs. In particular, once Beta secures funding for a container shipment, it provides a recipient with online access to its inventory database, and lets the recipient determine what to fill her container with. By allowing recipients themselves to pick what they are to receive, the recipient-driven model offers the potential to substantially reduce the mismatch between supply and demand. However, while this approach maximizes the container value for a given recipient at a given time, it remains unclear which recipient would be ideal to serve at each shipping opportunity. The sequence in which the recipients are served is important because MSROs often have low inventory levels for many critical products (e.g., biomedical equipment). Consequently, for a randomly selected recipient, her most-needed products may not be available in the MSRO's existing inventory, in which case she may fill her container with some products that are of secondary importance to her, while those products could be of primary importance to another recipient. Hence, appropriately determining which recipient to serve at each shipping opportunity is critical for better matching supply with demand. However, this is challenging as recipient needs are usually not known to the MSRO.

To improve MSROs' value provision capability, we identify implementable strategies to support MSROs' recipient selection decisions when recipient needs are private information. In particular, we propose a mechanism design approach where recipients are asked to report their preference rankings over a set of critical products that the MSRO has access to. The reported preference information is then used to determine which recipients to serve depending on what is available in MSRO inventory at each shipping opportunity. Motivated by Beta's best practice of providing recipients with online access to its inventory database, we first study a base model where the MSRO inventory levels are public information to the recipients. Our analysis shows that when inventory information is shared, recipients may have an incentive to misreport their preferences to improve the likelihood of being served, and the only truthful mechanism is random selection among recipients, which defeats the purpose of eliciting needs information.

Motivated by this result, we subsequently propose two operational strategies to improve MSROs' value provision capability: i) not sharing MSRO inventory information with recipients; and ii) withholding information regarding the preferences of other recipients in the recipient pool. To compare the MSRO's value provision under different settings characterized by combinations of the proposed operational strategies, we first show that any mechanism satisfying two natural properties called symmetry and acyclicity can be characterized by a score function. This score function assigns each recipient a score at each shipping opportunity based on her reported preference rankings and the available MSRO inventory, and then the recipient with a higher score is selected. We characterize the set of truthful score functions under each setting and show that, first, to ensure that recipients truthfully report their preferences, it is important for the MSRO to eliminate inventory information provision to recipients. Second, the total value provision to recipients can be further improved if the competitor information is eliminated, i.e., the MSRO should not provide a recipient with information regarding other recipients. We remark that these findings differ from those in many traditional supply chain management settings, where information sharing typically improves system performance [91, 92, 93].

We further investigate the value of cardinal mechanisms, where recipients are asked to report their valuations for different products, instead of just rankings. In contrast to what might be expected about the value of more granular information [94, 95, 96], we find that among a wide class of easy-to-implement mechanisms characterized by additive linear score functions, the ones that rely only on recipients' rankings of products perform equally well compared with those relying on valuations. This result is appealing from a practical perspective, as it suggests that there is no added value from further eliciting the exact valuations and it suffices for an MSRO to collect recipients' preference ranking information.

Finally, we develop a calibrated numerical study based on Beta's historical data to estimate the value of our proposed strategy consisting of a ranking-based scoring mechanism in conjunction with eliminating inventory and competitor information. We show that under this strategy, which involves only minimal change to the industry best practice, recipients can receive significantly more of their top-ranked products. Furthermore, this strategy closes more than 50% of the gap in value provision between the status quo and a clairvoyant solution which knows both recipients' valuations and future arrival volumes of different products. Due to its competitive performance and simplicity, the proposed strategy can be easily implemented in practice to help MSROs improve their value provision to recipients. Indeed, our findings already led to a change in Beta's practice for their allocation of biomedical equipment. Beta has now blinded the inventory of biomedical equipment in their online inventory system. They are currently working on adjusting their recipient application system to elicit recipients' preference rankings for different biomedical equipment, and implementing our scoring mechanism to support their recipient selection decisions.[1]

The rest of the paper is organized as follows. In §4.2, we provide an overview of the related literature and discuss our contributions. In §4.3, we present our model. In §4.4, we present the mechanism design problem and characterize the set of truthful mechanisms when the MSRO inventory information is provided to recipients. In §4.5, we show that in order to ensure that recipients truthfully reveal their needs, it is important to eliminate

[1]Medical supplies, which have higher inventory availability and are needed by virtually all recipients, continue to be managed as before.

the provision of MSRO inventory and competitor information to recipients. In §4.6, we discuss the value of cardinal mechanisms. Finally, we estimate the value of our proposed mechanism using Beta data in §4.7 and draw conclusions in §4.8.

4.2 Related Literature

Our work relates to several distinct streams of literature on medical surplus product allocation and resource allocation mechanisms.

Medical Surplus Recovery. First, our work contributes to the recently growing literature on medical surplus recovery and allocation. Case studies on MSROs suggest that the lack of operational expertise to match the uncertain supply with recipient demand is a major challenge for MSROs [97, 98, 99, 100]. Further, existing research in this area has focused on measuring recipients' medical surplus utilization rates, and empirically identified a significant mismatch between supply and demand [101, 102], largely driven by the fact that recipient needs are usually unknown to the MSRO. To our knowledge, [103] are the first to formally analyze a resource allocation model in this context. More specifically, [103] study a recipient-driven model where recipients are allowed to choose the content of the container they will be receiving, and show that this model can result in rushed container shipments and loss in value provision due to recipient competition. They suggest that switching to a provider-driven model will significantly improve the value provision to recipients. Yet, [103] assume known recipient valuations throughout the paper and do not address the problem of asymmetric information, which is the key challenge in this context. In contrast, we consider private recipient valuations and propose a mechanism design approach that enables the MSRO to determine which recipient to serve at each shipping opportunity based on revealed recipient preferences.

Resource Allocation in Humanitarian Context. Our work adds to the class of resource allocation problems in the broader humanitarian logistics field (see overviews in [104] and [105]). In this context, while many existing studies focus on optimal decision making in centralized settings [**lee2017combining**, e.g., 106, 107, 108, 109], a number of other studies consider incentive issues in decentralized systems [86, 110, 111, 112, 113, 114]. Among those, [86], who also consider a mechanism design approach, is most relevant to ours. In particular, [86] study a fleet size problem where the transportation needs of different humanitarian programs are private information. Our work is different from theirs in two aspects: i) while [86] consider the supply (the size of the fleet) as a decision, the supply of medical surplus is uncertain and uncontrollable; and ii) the key trade-off in their problem is to balance the overage and underage costs for a single resource, while we consider multiple product categories and the problem is to deliver the right product to the right recipient.

Optimal Mechanism Design for Resource Allocation. Our work also relates to resource allocation problems without monetary transfer and with sequential product arrivals, such as organ allocation and public housing. While most existing studies on organ allocation assume known recipient types [e.g., 115, 116, 117, 88, 89], one exception is [90], who study a kidney allocation problem where recipients report their risk types by joining one of the waitlists. However, in [90], patients are homogeneous in their product preferences (everybody prefers a higher quality kidney) and each patient receives at most one kidney. This is in contrast to the MSRO setting where recipients have heterogeneous product preferences and receive multiple items. There is also a growing literature that studies welfare-maximizing matching in public housing (or broader contexts) where items that arrive over time are matched with agents with private preference information [e.g., 118, 119]. In these studies, the focus is often on the effectiveness of different queuing disciplines (e.g., multiple waitlists); also, each agent typically receives at most one item (e.g., house), while we allow recipients to receive different product mixes.

Another approach for resource allocation without monetary transfer is to ask recipients to bid using tokens. For example, [120] considers such an approach for allocating food to food banks daily. However, [120] points out that "it is very rare to observe these kind of ‘monopoly money’ solutions being used to allocate resources in real world settings.” A key feature that makes this approach work in their setting is that a large number of recipients simultaneously participate in the game repeatedly on a daily basis, which minimizes strategic recipient behavior. This is in contrast to the MSRO setting where recipients around the world receive full-container shipments sequentially and infrequently (often only once for the foreseeable future for a given recipient), and only a limited number of recipients are served within a given time frame.

Our work also relates to rationing games in the literature [e.g., 121, 122, 123, 124]. In particular, these studies consider resource allocation problems where recipients/retailers have private information about their needs. A key difference is that while these studies mainly focus on the allocation of a single resource (e.g., production capacity) where retailers have the incentive to misreport their needs by exaggerating their orders, we study the allocation of multiple product categories where recipients’ strategic behavior is to misreport their preference rankings of different products.

Information Disclosure in Product Allocation and Supply Chain Management. Our results also relate to existing research on information disclosure/sharing in product allocation and supply chain management problems. First, in the product allocation literature, information disclosure has received extensive attention in auction problems (see [125] for a review). In these problems, the objective is to maximize the revenue of the auctioneer, and it has been shown that disclosing product (quality) information publicly to bidders often increases revenue [126]. This is well-known as the “linkage principle”. In this study, we extend the discussion of information disclosure to product allocation problems in a humanitarian context where there is no monetary transfer, and show that the disclosure of MSRO inventory information can lead to strategic misreporting of recipient preferences, which undermines the total value provision to recipients. We note that information disclosure has been studied less frequently in resource allocation problems without monetary transfers. For example, in organ allocation problems, due to the perishable nature of organs and that each recipient receives at most one organ, it is optimal to allocate each organ upon its arrival [90]; hence the discussion of inventory information is not relevant. In many static allocation problems such as school assignment, the inventory levels (e.g., the school capacities) are usually fixed and it suffices to ask participants to report their preferences among the available resources. However, in our setting, to support recipient selection decisions, it is important to ask recipients to report their preferences among the full set of products the MSRO has access to (which may or may not be available at the current period). We also note that our result differs from [103]'s discussion on MSRO inventory visibility, which suggests presenting lower-than-actual inventory levels to recipients in order to induce recipients to wait in a recipient-driven model. In contrast, we suggest eliminating inventory information provision in order to elicit truthful recipient needs information and support MSRO decision making towards recipient selection.

Second, in the supply chain management literature, a number of existing studies in traditional supply chain settings have shown that the sharing of (inventory) information typically improves system performance, for example, by enabling different locations to make more informed inventory replenishment decisions [e.g., 91, 92, 93]. In contrast, we show that sharing MSRO inventory information leads to a restricted set of truthful recipient selection mechanisms and hence limits the MSRO's value provision capability.

Cardinal vs. Ordinal Mechanisms. Finally, our work is also related to the literature that compares cardinal and ordinal mechanisms in resource allocation problems. In many contexts, each player has specific valuations for different products. If the allocation decisions are made based on reported player valuations, the mechanism is called cardinal [e.g., 127, 128]. Otherwise, if only the relative preference rankings are elicited, the mechanism is called ordinal [e.g., 129, 130]. In a school assignment model, [94] show that mechanisms eliciting cardinal preferences can do strictly better than those only eliciting ordinal information. In similar contexts, [95] and [96] further show that the ratio between the optimal social welfare of a cardinal and an ordinal mechanism can be arbitrarily large. However, the above work either demonstrates the value of cardinal information assuming it is given a priori or considers static allocation in a large market where each participant is a price taker (hence strategic behavior is minimized). Unlike the above studies, we consider a finite market and determine which recipient to serve at each period. In this case, as described in more detail in Section §4.6, we find zero value added from a cardinal mechanism over an ordinal mechanism under a wide class of implementable mechanisms.

4.3 Model Setup

In our base model, we consider a setup where an MSRO allocates products to two recipients in two periods ($t = 0, 1$). We present a multi-recipient extension in the appendix. Below, we describe model components in detail and summarize key notation in Table 4.1.

Table 4.1: Summary of key notation.

Section	Notation	Description
§4.3	$v_{i,j}$	Valuation of recipient i for product $j, i = 1, 2; j = 1, ..., N$
	$\gamma_{i,j}$	Ranking of recipient i for product $j, i = 1, 2; j = 1, ..., N$
	$\phi_{-i,i}$	Probability density function (p.d.f.) for recipient $-i$'s belief on $\boldsymbol{v}_i, i = 1, 2$
	μ_j	Inventory level of product j at $t = 0, j = 1, ..., N$
	ϵ_j	Amount of new arrivals of product j at $t = 1, j = 1, ..., N$
	K	Total amount of products each recipient receives in one shipment
	$y^t_{i,j}$	Amount of product j recipient i receives if she is served at $t = 0, 1, i = 1, 2; j = 1, ..., N$
	δ	Time discount factor the MSRO uses in measuring total value provision
§4.4	$p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$	Probability that recipient 1 is selected given reported rankings $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$
	$x^t_{i,j}$	Expected amount of product j recipient i receives at $t = 0, 1, i = 1, 2; j = 1, ..., N$
	$\Pi^t_i(\boldsymbol{v}_i, \boldsymbol{\gamma}'_i)$	Expected payoff of recipient i at $t = 0, 1$ with true valuation $\boldsymbol{v}_i$ and reported ranking $\boldsymbol{\gamma}'_i$
	$\phi_{0,i}$	Probability density function (p.d.f.) for MSRO's belief on $\boldsymbol{v}_i, i = 1, 2$
§4.5	ψ_i	Probability mass function (p.m.f.) for recipient i's belief on $\boldsymbol{\mu}, i = 1, 2$
	$\rho_{i,j}$	Index of recipient i's rank-j product, $i = 1, 2; j = 1, ..., N$.
	g_j	Points each recipient gets for each unit of their rank-j product in the MSRO inventory

Valuations and Rankings. In line with practice, we assume that different recipients, which typically represent healthcare facilities with different specializations from various regions, have heterogeneous valuations on products. In particular, let $v_{i,j}$ denote the valuation of recipient i for product $j, i = 1, 2; j = 1, ..., N$. Similarly, let $\gamma_{i,j}$ denote the ranking of recipient i for product $j, i = 1, 2; j = 1, ..., N$, where a smaller number indicates a higher preference (e.g., the most preferred product has ranking 1). Let $\boldsymbol{v}_i \doteq (v_{i,1}, ..., v_{i,N})$ and $\boldsymbol{\gamma}_i \doteq (\gamma_{i,1}, ..., \gamma_{i,N})$. We assume that recipients' preferences are strict, i.e., $v_{i,j} \neq v_{i,j'}, \forall j \neq j'$.[2] Then, for any given valuation $\boldsymbol{v}_i$, there exists a unique ranking $\boldsymbol{\gamma}_i$, which is a permutation of $1, ..., N$, such that for any $j \neq j'$, $v_{i,j} > v_{i,j'}$ if and only if $\gamma_{i,j} < \gamma_{i,j'}$. Let $V \doteq \{\boldsymbol{v}_i \in \mathbb{R}^N : v_{i,j} \geq 0, j = 1, ..., N; v_{i,j} \neq v_{i,j'}, \forall j \neq j'\}$, and let Γ be the set of all possible rankings $\boldsymbol{\gamma}_i$. In line with practice, we assume that $\boldsymbol{v}_i$ is private information of recipient $i, i = 1, 2$. Let $-i$ denote the other recipient, and we assume that recipient $-i$ has a belief on $\boldsymbol{v}_i$ with joint probability density function (p.d.f.) $\phi_{-i,i}(\boldsymbol{v}_i) > 0, \forall \boldsymbol{v}_i \in V$.

Inventory Levels. Let $\boldsymbol{\mu} = (\mu_1, ..., \mu_N)$ denote the MSRO inventory levels of products $1, ..., N$ at $t = 0$, and let $\boldsymbol{\epsilon} = (\epsilon_1, ..., \epsilon_N)$ denote the amount of random arrivals of products $1, ..., N$ at $t = 1$ with known distributions and finite means. We consider indivisible products, i.e., μ_j's and ϵ_j's are integer numbers. We assume that the total quantity of products each recipient receives is the same and is equal to K, which represents the MSRO's total quantity limit placed on each recipient, the container capacity or a portion of the container capacity dedicated for the product categories under consideration. Then, without loss of generality, we assume that $\mu_j \leq K, j = 1, ..., K$. To ensure that at least one recipient can be served at $t = 0$, we assume $\sum_{j=1}^{N} \mu_j \geq K$. Finally, in line with Beta's existing practice, we assume that $\boldsymbol{\mu}$ is known to recipients in our base model.

Recipient Selection. In practice, based on multiple factors such as country-wise healthcare needs, donor preferences and available funding for container shipments, a few recipients from the recipient pool will be considered "eligible" for receiving a container shipment. Each eligible recipient will receive one container shipment and will be served se-

[2]Considering non-strict preferences leads to a more complicated technical exposition without adding new insights.

quentially over time because of the tight operational constraints faced by MSROs. To mimic this process, we assume that the two recipients in our model (assuming both are eligible from $t = 0$) are served one by one in the two respective periods, each receiving one container shipment. As briefly discussed in §4.1, since the MSRO inventory levels dynamically evolve over time, the total value provision to recipients can be quite different based on the sequence in which recipients are served. Hence, in this paper, we focus on determining which recipient to serve at each shipping opportunity.

Once a recipient is selected, in line with Beta's current practice, we assume that the recipient receives the best bundle (i.e., the set of items that maximizes its container value) from the available inventory. This is appealing from both implementation and analytical tractability perspectives. We discuss the implications of relaxing this assumption in §4.8 and in the appendix.

We next formulate the MSRO's recipient selection problem that maximizes the total value provision when recipient needs are known to the MSRO. For $i = 1, 2$ and $j = 1, ..., N$, let $y_{i,j}^0$ and $y_{i,j}^1$ be the amount of product j recipient i receives if she is served at $t = 0$ and $t = 1$, respectively, and let π_i^0 and π_i^1 be the value provision to recipient i if she is served at $t = 0$ and $t = 1$, respectively. Then, if the valuations ($v_{i,j}$'s) were known, the MSRO could determine π_i^0 and π_i^1 according to:

$$\pi_i^0 \doteq \max_{y_{i,j}^0} \sum_{j=1}^{N} v_{i,j} y_{i,j}^0, \tag{4.1}$$

$$\text{s.t. } y_{i,j}^0 \leq \mu_j, j = 1, ..., N; \sum_{j=1}^{N} y_{i,j}^0 \leq K.$$

$$\pi_i^1 \doteq \max_{y_{i,j}^1} \sum_{j=1}^{N} v_{i,j} y_{i,j}^1, \tag{4.2}$$

$$\text{s.t. } y_{i,j}^1 \leq \mu_j - y_{-i,j}^0 + \epsilon_j, j = 1, ..., N; \sum_{j=1}^{N} y_{i,j}^1 \leq K.$$

Given π_i^0 and π_i^1, the MSRO selects a recipient at $t = 0$ (while the other recipient is served at $t = 1$) that maximizes the expected total discounted value provision, π_{OPT}, according to:

$$\pi_{OPT} \doteq \max_{i=1,2}(\pi_i^0 + \delta E_{\boldsymbol{\epsilon}}[\pi_{-i}^1]), \tag{4.3}$$

where $\delta \in [0, 1)$ denotes the time discount factor the MSRO uses to discount the value provided to recipients at $t = 1$, and $E_{\boldsymbol{\epsilon}}$ denotes the expectation over $\boldsymbol{\epsilon}$.

Clearly, π_{OPT} and the optimal solution to Equation 4.3 can be easily determined if recipient valuations were known to the MSRO. However, as discussed above, this is not the case in practice. Therefore, we next present a mechanism design approach to help the MSRO elicit recipient needs information and determine which recipient to serve when recipient needs are not known.

4.4 The Mechanism Design Problem

We focus our attention on truthful direct mechanisms (in which players truthfully report their types) because thanks to the well-known revelation principle, any equilibrium outcome that can be implemented by an arbitrary mechanism can also be implemented by a truthful direct mechanism [131]. Furthermore, a truthful mechanism has several advantages from a practical perspective as it eliminates strategic behavior of recipients as well as the concern that naive or honest recipients will be at a disadvantage compared to those who are more strategic.

In particular, we focus on ordinal mechanisms where recipients only report their preference ranking $\boldsymbol{\gamma}_i$, since an ordinal mechanism is simple and and has been successfully implemented in several other settings such as voting and school assignment [**carroll2017mechanisms**]. We study cardinal mechanisms in §4.6. As in the known recipient needs case described in §4.3, we assume that one recipient is served at each period, and when served, the recipient receives the best bundle (based on its reported rankings) from the available inventory. The problem is then to determine which recipient to serve at each period. Then, an ordinal mechanism is defined as a function $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$ that specifies which recipient to serve at $t = 0$ under any given ranking profile $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$.

We first consider two deterministic priority rules for ease of implementation: serving recipient 1 ($p = 1$) and serving recipient 2 ($p = 0$). We also consider a third option of serving a random recipient ($p = \frac{1}{2}$) to break a tie (e.g., when the two recipients report the same ranking). Then, a mechanism is a function $p : \Gamma^2 \to \{0, \frac{1}{2}, 1\}$. The sequence of events is as follows: i) the MSRO announces a mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$ at the beginning of $t = 0$; ii) each recipient i reports her ranking $\boldsymbol{\gamma}_i$; and iii) based on the reported rankings and the pre-announced mechanism, the MSRO determines which recipient to serve at $t = 0$ (while the other recipient is served at $t = 1$). We note that we do not ask recipients to specify the quantity they need for each product because in practice, recipeints' needs for critical products are often truncated by MSROs' inventory levels (or quantity limits MSROs impose for each recipient and each product). For example, Beta usually sends only one or at most two pieces of each biomedical equipment to a given recipient. For cases with higher inventory levels and when there is a significantly diminishing return in recipients' valuations, our model can be modified by treating an additional unit of each product as a different product.

Next, we introduce additional notation which is used for defining the truthfulness of a mechanism. Recall that the optimal solutions $y^0_{i,j}$ and $y^1_{i,j}$ to Equations 4.1 and 4.2 define the amount of product j recipient i receives if she is served at $t = 0$ and $t = 1$, respectively, and that $y^0_{i,j}$ and $y^1_{i,j}$ can be determined solely based on recipients' reported rankings $(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i})$. Therefore, we denote $y^0_{i,j}$ and $y^1_{i,j}$ as $y^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i})$ and $y^1_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i})$, respectively. Let $p_1 \doteq p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$, $p_2 \doteq 1 - p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$, $x^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}) \doteq p_i y^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i})$, and $x^1_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}) \doteq (1 - p_i)E_{\boldsymbol{\epsilon}}[y^1_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i})]$. Then, $x^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i})$ and $x^1_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i})$ denote the expected amount of product j recipient i receives at $t = 0$ and $t = 1$, respectively. For each recipient i, given her true valuation $\boldsymbol{v}_i$ and reported ranking $\boldsymbol{\gamma}'_i$, her expected payoff at period t under a given mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$ is $\Pi^t_i(\boldsymbol{v}_i, \boldsymbol{\gamma}'_i) \doteq E_{\boldsymbol{\gamma}_{-i}}[\sum_{j=1}^N v_{i,j} x^t_{i,j}(\boldsymbol{\gamma}'_i, \boldsymbol{\gamma}_{-i})], t = 0, 1$, where $E_{\boldsymbol{\gamma}_{-i}}$

denotes the expectation over recipient i's belief on recipient $-i$'s ranking $\boldsymbol{\gamma}_{-i}$.

Definition 3 *An ordinal mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$ is truthful if for each recipient i with any valuation $\boldsymbol{v}_i \in V$, the payoff (Π_i^0, Π_i^1) obtained from reporting her true ranking $\boldsymbol{\gamma}_i = \boldsymbol{\gamma}_i(\boldsymbol{v}_i)$ is not lexicographically dominated[3] by that of reporting any other ranking $\boldsymbol{\gamma}_i' \in \Gamma$, i.e., $\forall i, \boldsymbol{v}_i \in V, \boldsymbol{\gamma}_i' \in \Gamma$,*

$$\textit{i)}\ \Pi_i^0(\boldsymbol{v}_i, \boldsymbol{\gamma}_i) > \Pi_i^0(\boldsymbol{v}_i, \boldsymbol{\gamma}_i');\ \textit{or}\ \ \textit{ii)}\ \Pi_i^0(\boldsymbol{v}_i, \boldsymbol{\gamma}_i) = \Pi_i^0(\boldsymbol{v}_i, \boldsymbol{\gamma}_i')\ \textit{and}\ \Pi_i^1(\boldsymbol{v}_i, \boldsymbol{\gamma}_i) \geq \Pi_i^1(\boldsymbol{v}_i, \boldsymbol{\gamma}_i'). \tag{4.4}$$

The truthfulness notion in Definition 3 is based on the lexicographical order, which has been widely used in the mechanism design literature for comparison of multi-dimensional vectors [132]. In our setting, it captures the observation that recipients are usually impatient for critical resources and compete for the current shipping opportunity [103]. We also consider an alternative definition of truthfulness in the appendix based on a discounted payoff model and show that our key insights continue to hold under that definition.

Finally, we allow the MSRO to have different beliefs on recipients' valuations compared with recipients' beliefs on each other's valuations, because the MSRO may have specific information about the recipients (e.g., geographic regions, types of hospitals, etc.), and they may or may not share such information with recipients. For $i = 1, 2$, let $\phi_{0,i}$ denote the p.d.f. of the MSRO's belief on $\boldsymbol{v}_i$. Then, an optimal truthful mechanism can be found by solving:

$$\max_{p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)} E_{\boldsymbol{v}_1, \boldsymbol{v}_2}\left[\sum_{i=1}^{2}\left(\sum_{j=1}^{N} v_{i,j} x_{i,j}^0\big(\boldsymbol{\gamma}_i(\boldsymbol{v}_i), \boldsymbol{\gamma}_{-i}(\boldsymbol{v}_{-i})\big) + \delta \sum_{j=1}^{N} v_{i,j} x_{i,j}^1\big(\boldsymbol{\gamma}_i(\boldsymbol{v}_i), \boldsymbol{\gamma}_{-i}(\boldsymbol{v}_{-i})\big)\right)\right], \tag{4.5}$$

subject to the truthfulness constraints characterized by Inequality 4.4 and that $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2) \in \{0, \frac{1}{2}, 1\}$, where the expectation is taken over the MSRO's beliefs $\phi_{0,i}$ on recipients' valuations $\boldsymbol{v}_i, i = 1, 2$, and $\boldsymbol{\gamma}_i(\boldsymbol{v}_i)$ denotes the ranking vector that corresponds to $\boldsymbol{v}_i, i = 1, 2$.

[3] A vector $\boldsymbol{x} = (x_1, x_2)$ is said to be lexicographically dominated by another vector $\boldsymbol{x}' = (x_1', x_2')$ if i) $x_1 < x_1'$ or ii) $x_1 = x_1'$ and $x_2 < x_2'$.

4.4.1 Characterization of Truthful Mechanisms

Intuitively, a desirable property of a mechanism that achieves high total value provision is that the probability that each recipient is selected at $t = 0$ increases when her needs are better met with the available inventory. However, in the following proposition, we show that in order to ensure that recipients truthfully report their needs, the mechanism should be such that each recipient assigns a fixed probability to the event of being selected no matter what her ranking is.

Proposition 9 *Suppose $\boldsymbol{\mu}$ is public information. If a mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$ is truthful, then the probability each recipient assigns to the event of being selected at $t = 0$ based on her belief on the other recipient's rankings is a constant independent of what she reports, i.e., there exists $\tilde{p} \in [0, 1]$ such that $E_{\boldsymbol{\gamma}_2}[p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)] = \tilde{p}, \forall \boldsymbol{\gamma}_1 \in \Gamma$, and $E_{\boldsymbol{\gamma}_1}[1 - p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)] = 1 - \tilde{p}, \forall \boldsymbol{\gamma}_2 \in \Gamma$.*

Proposition 9 says that to ensure truth telling, the probability each recipient assigns to being selected at $t = 0$ cannot depend on what she reports. In other words, the MSRO can only rely on its own belief on recipients' preferences to determine the allocation sequence. This result is to some extent disappointing as it defeats the purpose of eliciting preference information. The intuition here is that if reporting different rankings results in different probabilities of being selected, a recipient can try to obtain a better chance to be selected by misreporting her rankings. More importantly, the incentive for recipients to misreport stems from the fact that recipients know the MSRO inventory levels. Suppose that the mechanism prioritizes a recipient who can benefit more from the available inventory. Then, recipients knowing the MSRO inventory levels may have an incentive to claim that they have higher rankings for the products with high inventory levels, so as to have a better chance of being selected (a better chance of being selected implies a higher expected payoff Π_i^0 if the recipient's valuations of different products are sufficiently close to each other).

Inspired by the above result, we next compare two cases where the MSRO inventory information is and is not provided to recipients, respectively. We first show that any mechanism that satisfies two natural properties, symmetry and acyclicity (defined in §4.5), can be characterized by a score function. Then, we characterize the set of truthful score functions under each setting and show that the total value provision to recipients can be improved when inventory information provision is eliminated, but the extent of the improvement depends on recipients' beliefs on each other's rankings. We quantify the value of eliminating inventory and competitor information by comparing the cases where inventory and/or competitor information is and is not provided, respectively.

4.5 Value of Eliminating Inventory and Competitor Information

When the MSRO inventory information is not provided to recipients, we assume that each recipient i has a belief on the inventory levels $\boldsymbol{\mu} = (\mu_1, ..., \mu_N)$. In particular, let $M \doteq \{\boldsymbol{\mu} \in \mathbb{Z}^N : 0 \leq \mu_j \leq K, j = 1, ..., N; \sum_{j=1}^{N} \mu_j \geq K\}$ be the set of all possible inventory levels. Assume that each recipient i believes that there is a probability $\psi_i(\boldsymbol{\mu})$ for the true inventory levels to be $\boldsymbol{\mu}$, where $\sum_{\boldsymbol{\mu} \in M} \psi_i(\boldsymbol{\mu}) = 1$. Note that this definition captures both cases where inventory information is and is not provided to recipients (for the case where inventory information is provided to recipients, ψ_i is simply a deterministic distribution with probability one attributed to the true inventory levels).

Recall that the MSRO's problem is to determine which recipient to serve at each period based on recipients' reported rankings. When the MSRO inventory information is not provided to recipients, the MSRO needs to specify the decision rule under all possible inventory levels. Then, a mechanism is a function of not only the reported rankings $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$, but also the inventory levels $\boldsymbol{\mu}$, denoted by $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$, where recipients take expectation over $\boldsymbol{\mu}$ to evaluate their expected payoffs.

Finding an optimal mechanism where monetary transfer is not allowed and recipients have multi-dimensional valuations is notoriously difficult [128]. To address this challenge, we next show that in our setting, the structure of a mechanism can be significantly simplified under two natural properties, symmetry and acyclicity. More specifically, we show that any mechanism satisfying these two properties can be characterized by a score function. This provides an intuitive representation of the mechanism and a path for identifying implementable approaches for practice.

Definition 4 *A mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$ is symmetric in recipients if for any $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \Gamma$ and $\boldsymbol{\mu} \in M$, $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu}) = 1 - p(\boldsymbol{\gamma}_2, \boldsymbol{\gamma}_1, \boldsymbol{\mu})$, i.e., if the rankings of the two recipients are exchanged, the probability for each recipient to be selected would also be exchanged.*

Definition 5 *A mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$ is acyclic if for any fixed $\boldsymbol{\mu}$, $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu}) \geq \frac{1}{2}$ and $p(\boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3, \boldsymbol{\mu}) \geq \frac{1}{2}$ imply $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_3, \boldsymbol{\mu}) \geq \frac{1}{2}$, where $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ denote three ranking vectors, and the last inequality holds as an equality if and only if the first two inequalities are both equalities.*

Symmetry in players has been widely considered in the mechanism design literature to ensure equal treatment of equals [129]. In our context, it means the allocation probability depends only on the reported rankings of the recipients but not on the specific indices of the recipients. Acyclicity is also intuitive and widely considered in the literature [133]. In our context, it implies that if recipient 1 is more likely to be selected than recipient 2 when they report $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$, respectively, and when they report $\boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}_3$, respectively, then recipient 1 is more likely to be selected than recipient 2 when they report $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_3$, respectively.

Next, we show that any symmetric and acyclic mechanism is characterized by a *score function*, where each recipient is assigned a score based on her reported rankings and the true inventory levels. If the scores of the two recipients are equal, then each recipient is selected with an equal probability; otherwise, the recipient with a higher score is selected.

Proposition 10 *A mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$ is symmetric and acyclic if and only if there exists a score function $g : \Gamma \times M \to \mathbb{R}$ such that the score for each recipient i is $s_i =$*

$g(\boldsymbol{\gamma}_i, \boldsymbol{\mu})$, *and:*

$$p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu}) = \begin{cases} 1 & \textit{if } s_1 > s_2, \\ 0 & \textit{if } s_1 < s_2, \\ \frac{1}{2}, & \textit{if } s_1 = s_2. \end{cases} \tag{4.6}$$

With the above result, to find a mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$, it suffices to find a score function $s_i = g(\boldsymbol{\gamma}_i, \boldsymbol{\mu})$. In particular, unlike in a general mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$ where one needs to specify which recipient to serve under all possible combinations of ranking profiles, the score of each recipient depends only on her own but not the other recipient's rankings, which is intuitive, easy to implement, and enables a multi-recipient extension of our results (see appendix). If we further require symmetry in products, i.e., the allocation probability depends only on the inventory levels of different products but not on the product indices, then it can be proven that Proposition 10 holds with an even simpler score function $g(\mu_{\rho_{i,1}}, ..., \mu_{\rho_{i,N}})$, where $\rho_{i,j}$ denotes the product index for which recipient i's ranking is j. Finally, we remark that while acyclicity is essential for ensuring a score function representation, the main insights in this section continue to hold if the symmetry assumption is relaxed by considering recipient-specific score functions (see the appendix for details).

Scoring-based rules are simple, easy to communicate and commonly used in a number of other settings such as voting systems and organ allocation problems [134, 89]. For example, in liver allocation, priorities for patients on the liver transplant waitlist are determined by the Model for End-Stage Liver Disease (MELD) scoring system [135]. Similarly, in kidney allocation, the prioritization of patients is usually determined by the Kidney Allocation Score (KAS), which consists of a weighted additive sum of several score components that are functions of patient and/or organ characteristics [89]. In our setting, a simple and intuitive score function is an additive linear function of the MSRO inventory levels:

$$s_i = \sum_{j=1}^{N} g_j \mu_{\rho_{i,j}}, i = 1, 2. \tag{4.7}$$

Under such a score function, each recipient i gets g_j points for each unit of her rank-j product in inventory. For example, when $g_1 \geq ... \geq g_N$, a recipient is assigned a higher score if there is a higher inventory of her top-ranked products in the system.[4] Intuitively, such a score function favors the selection of a recipient whose needs are better met by the available inventory. The determination of a mechanism of this structure reduces to determining N parameters $g_j, j = 1, ..., N$, which is simpler both computationally and implementation-wise compared to the general score function $s_i = g(\boldsymbol{\gamma}_i, \boldsymbol{\mu})$ of dimension $N! \times (K+1)^N$ (or $(K+1)^N$ with symmetry in products).

In the remainder of this section, we focus on an additive linear score function to quantify the value of eliminating inventory information. We note that additive linearity is not a particularly restrictive assumption in our setting as it is in line with the value provision equations defined in Equations 4.1 and 4.2. Further, we show that the set of truthful mechanisms for the case where inventory information is provided (i.e., Proposition 11) remains the same under general score functions, hence our characterization of the difference between the cases where inventory information is and is not provided allows us to obtain a lower bound on the true value of eliminating inventory information. We characterize the set of truthful additive linear score functions for both cases where inventory information is and is not provided to recipients in §5.1 and §5.2, respectively.

4.5.1 With Inventory Information

We first analyze the case where the MSRO shares its inventory information with recipients (we denote this case by subscript I). In this case, each recipient i's belief on the inventory levels consists of a unit mass on the true inventory vector $\boldsymbol{\mu}$, i.e., $\psi_i(\boldsymbol{\mu}) = 1, i = 1, 2$. Then, truthfulness of a mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$ is defined in the same way as that in Definition 3, and we say that a score function is truthful if the associated mechanism defined in Equation

[4] In voting systems, a similar scoring method called Borda count has been widely considered [134]. The key difference is that in voting systems a score is assigned to each candidate and used as a measure of the popularity of candidates, while we use a score to measure the fit between the MSRO inventory levels and recipients' needs.

4.6 is truthful.

Recall that in Proposition 9, we have shown that with inventory information, the probability each recipient assigns to being selected cannot depend on what she reports. This result becomes even more intuitive when we consider mechanisms that are characterized by score functions.

Proposition 11 *For case I, the score of each recipient cannot depend on recipients' reported rankings. In particular, if a score function $s_i = \sum_{j=1}^{N} g_j \mu_{\rho_{i,j}}, i = 1, 2$ is truthful, then it can be characterized by any g_j's such that $g_1 = ... = g_N$.*

It follows from Proposition 11 that $s_1 = s_2$, and therefore $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu}) = \frac{1}{2}, \forall \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \Gamma$. Proposition 11 suggests that when inventory information is provided to recipients, the score of each recipient has to be the same constant, and the mechanism reduces to random selection among recipients. Furthermore, this result continues to hold under general score functions (see appendix). That is, the only truthful symmetric and acyclic mechanism is random selection among recipients. Yet, a randomized selection policy which does not consider the needs of recipients can lead to a significant value loss. In the following subsection, we show that eliminating the inventory information enlarges the set of truthful mechanisms, leading to an improvement in value provision.

4.5.2 Without Inventory Information

We next analyze the case where the MSRO does not share its inventory information with recipients (we denote this case by NI). In this case, first, we assume that each recipient assigns a positive probability to all possible inventory levels, i.e., $\psi_i(\boldsymbol{\mu}) > 0, \forall \boldsymbol{\mu} \in M$. Second, since we focus on a set of critical products (e.g., biomedical equipment) which typically all have relatively small donation volumes, it is usually difficult to tell which product has a higher inventory level than others. Hence, we assume that recipients' belief ψ_i is symmetric in products, i.e., the probability mass does not change under any permutation of a given inventory vector.[5] Formally, for any $\boldsymbol{\mu}, \boldsymbol{\mu}' \in M$, if $\mu_j = \mu'_k, \mu_k = \mu'_j$ for some $j, k = 1, ..., N$, and $\mu_l = \mu'_l, \forall l \neq j, k$, then $\psi_i(\boldsymbol{\mu}) = \psi_i(\boldsymbol{\mu}')$. Note that if the distributions of the inventory levels of different products are independent, then symmetry simply implies that the distributions of the inventory levels of different products are identical.

Let Ψ be the set of beliefs ψ_i that satisfy the above two properties. We next define truthfulness of a mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$ for case NI. For that, we need to define payoff functions under all possible inventory levels, where recipients take expectation over $\boldsymbol{\mu}$ to evaluate their expected payoffs. In particular, we denote $y^0_{i,j}$ and $y^1_{i,j}$ as $y^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})$ and $y^1_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})$, respectively. Let $p_1 \doteq p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$, $p_2 \doteq 1 - p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$, $x^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu}) \doteq p_i y^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})$, and $x^1_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu}) \doteq (1-p_i) E_{\boldsymbol{\epsilon}}[y^1_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})]$. For each recipient i, given her true valuation $\boldsymbol{v}_i$, reported ranking $\boldsymbol{\gamma}'_i$ and the MSRO inventory levels $\boldsymbol{\mu}$, her expected payoff at t is $\Pi^t_i(\boldsymbol{v}_i, \boldsymbol{\gamma}'_i, \boldsymbol{\mu}) \doteq E_{\boldsymbol{\gamma}_{-i}}[\sum_{j=1}^N v_{ij} x^t_{i,j}(\boldsymbol{\gamma}'_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})]$, $t = 0, 1$. Then, truthfulness for case NI is defined as follows.

Definition 6 *For the case without inventory information, an ordinal mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$ is truthful if for each recipient i with valuation $\boldsymbol{v}_i \in V$ and symmetric belief $\psi_i \in \Psi$ on the inventory vector, the expected payoff obtained from reporting her true ranking $\boldsymbol{\gamma}_i = \boldsymbol{\gamma}_i(\boldsymbol{v}_i)$ is not lexicographically dominated by that of reporting any other ranking $\boldsymbol{\gamma}'_i \in \Gamma$, i.e., $\forall i, \boldsymbol{v}_i \in V, \psi_i \in \Psi, \boldsymbol{\gamma}'_i \in \Gamma$,*

$$i) E_{\boldsymbol{\mu}}[\Pi^0_i(\boldsymbol{v}_i, \boldsymbol{\gamma}_i, \boldsymbol{\mu})] > E_{\boldsymbol{\mu}}[\Pi^0_i(\boldsymbol{v}_i, \boldsymbol{\gamma}'_i, \boldsymbol{\mu})]; \text{ or}$$

$$ii)\ E_{\boldsymbol{\mu}}[\Pi^0_i(\boldsymbol{v}_i, \boldsymbol{\gamma}_i, \boldsymbol{\mu})] = E_{\boldsymbol{\mu}}[\Pi^0_i(\boldsymbol{v}_i, \boldsymbol{\gamma}'_i, \boldsymbol{\mu})] \text{ and } E_{\boldsymbol{\mu}}[\Pi^1_i(\boldsymbol{v}_i, \boldsymbol{\gamma}_i, \boldsymbol{\mu})] \geq E_{\boldsymbol{\mu}}[\Pi^1_i(\boldsymbol{v}_i, \boldsymbol{\gamma}'_i, \boldsymbol{\mu})].$$

Note that the truthfulness notion in Definition 6 is "robust" to recipients' belief ψ_i in the sense that we require that truth-reporting is better off for recipients with any symmetric belief $\psi_i \in \Psi$.

[5] For the case where the product set contains products with substantially different inventory levels, one can divide the product set into different tiers such that the symmetric assumption is reasonable within each tier. Then, our framework can be applied by asking recipients to report their rankings within each tier.

Since the only truthful score function for the case with inventory information is random selection among recipients, it is straightforward to show that the set of truthful score functions after eliminating inventory information is at least as large as before. Let π_I and π_{NI} be the total value provision of an optimal truthful additive linear score function (i.e., a truthful additive linear score function that maximizes Equation 4.5) for cases I and NI, respectively. Then, we have the following result. We remark that the same conclusion holds under general score functions.

Theorem 4 $\pi_{NI} \geq \pi_I$.

Theorem 4 says that the value of eliminating inventory information is always non-negative. The basic intuition is that when the MSRO inventory information is not provided to recipients, recipients' incentive to claim that they like the products with high inventory levels is eliminated, and hence the MSRO is able to use the revealed preferences to determine which recipient to serve. This result has an important practical implication as it suggests a different operational strategy regarding the disclosure of MSRO inventory information compared with the industry best practice implemented by Beta (where recipients have online access to Beta's inventory database and can observe its inventory levels). In order to truthfully elicit recipients' preference information and use that information to support the MSRO's recipient selection decision, it is necessary to ask recipients to report their rankings without disclosing the MSRO inventory information.

In order to quantify the value of eliminating inventory information (i.e., the gap between π_{NI} and π_I), we next characterize the set of truthful additive linear score functions for case NI. To do so, we note that the set of truthful mechanisms for this case depends on recipients' beliefs on each other's rankings (while our result in Proposition 11 for case I holds under any recipients' beliefs on each other's rankings). In that regard, we consider the following two sub-cases: i) a recipient believes that the other recipient has a specific ranking with a sufficiently large probability (§4.5.2); and ii) both recipients believe that it is equally likely for the other recipient to have any ranking (§4.5.2). We refer to these two cases as with and without competitor information, respectively, and we particularly focus on these two cases as they correspond to the smallest and largest set of truthful mechanisms, respectively, among all possible beliefs $\phi_{i,-i}, i = 1, 2$ (as Proposition 14 will demonstrate). By comparing to case I, the value provision improvement of these two cases provide a lower and an upper bound, respectively, on the value of eliminating inventory information.

With Competitor Information

We first analyze the case where the MSRO shares the information regarding other recipients in the recipient pool (e.g., recipients' identities, geographic regions, types of hospitals, etc.; we denote this case by NI-C). In this case, we assume that the information provided by the MSRO allows recipients to form firm beliefs about other recipients' preferences. Specifically, we consider a scenario where recipient 1 believes that recipient 2 has a specific ranking with a large probability, while recipient 2's belief can be arbitrary (i.e., our result holds as long as one recipient has firm beliefs). Given recipient 1's belief $\phi_{1,2}$ on $\boldsymbol{v}_2$, let $P_{\phi_{1,2}}(\boldsymbol{\gamma}_2)$ denote the probability that recipient 1 believes that recipient 2's ranking is $\boldsymbol{\gamma}_2$. Then, we assume that there exists $\boldsymbol{\gamma}_2 \in \Gamma$ such that $P_{\phi_{1,2}}(\boldsymbol{\gamma}_2) \geq 1 - \theta$, where θ is a small number.[6]

Proposition 12 *For case NI-C, a score function $s_i = \sum_{j=1}^{N} g_j \mu_{\rho_{i,j}}, i = 1, 2$ is truthful if and only if $g_1 \geq g_2 = ... = g_N$, or $g_1 = ... = g_{N-1} \geq g_N$.*

The set of truthful mechanisms characterized in Proposition 12 is strictly larger than that characterized in Proposition 11. Proposition 12 says that in case NI-C, the score a recipient gets for each unit of her rank-1 (or rank-N) product can be different from those for other products, but the unit score for all other products have to be the same. The intuition is as follows: If $g_1 > g_2 = ... = g_N$, then the mechanism is such that the recipient whose rank-1 product has a higher inventory level is selected. Under this mechanism, misreporting

[6]More specifically, $0 \leq \theta < \max\{\frac{1}{N}, \frac{1}{3}\}$.

the rank-1 product increases the probability of being selected when the *reported* rank-1 product is available (or has a high inventory level), but decreases the probability of being selected when the *true* rank-1 product is available. Then intuitively, recipients do not have an incentive to misreport their rank-1 products. In contrast, if the mechanism uses the full ranking information (i.e., $g_1 > ... > g_N$), then a recipient who is informed about the other recipient's rankings may have an incentive to misreport. For example, under some beliefs on inventory levels, if a recipient's rank-1 and rank-2 products are the other recipient's rank-N and rank-1 products, respectively, then the first recipient may want to reverse the reported rankings of her true rank-1 and rank-2 products so as to obtain a larger probability of being selected when her true rank-2 product is available, without decreasing the probability of being selected when her true rank-1 product is available. The proof of Proposition 12 formalizes this argument.

In sum, Proposition 12 shows that there is value added from eliminating inventory information, but the set of truthful mechanisms remains relatively small when recipients are informed about each other's rankings, which implies limited value added from only eliminating inventory information. In the following subsection, we show that eliminating competitor information along with inventory information leads to a substantially larger set of truthful mechanisms, and hence a further improvement in the MSRO's value provision capability.

Without Competitor Information

We next analyze the case where the MSRO withholds information regarding other recipients in the recipient pool (we denote this case by NI-NC). In this case, we assume that recipients have virtually no information about their competitor's identities or preferences, and that they have a prior belief under which their competitor can have any ranking with an equal probability. That is, $P_{\phi_{1,2}}(\boldsymbol{\gamma}_2) = P_{\phi_{2,1}}(\boldsymbol{\gamma}_1) = \frac{1}{N!}, \forall \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \Gamma$. This assumption is reasonable when the MSRO serves large recipient bases and different recipients have diverse needs (recall that we consider a set of critical products such as biomedical equipment).

Proposition 13 *For case $NI\text{-}NC$, a score function $s_i = \sum_{j=1}^{N} g_j \mu_{\rho_{i,j}}, i = 1, 2$ is truthful if and only if $g_1 \geq ... \geq g_N$.*

The set of truthful mechanisms characterized in Proposition 13 is strictly larger than those characterized in Propositions 11 and 12. More specifically, Proposition 13 says that after both inventory and competitor information provision are eliminated, the set of truthful mechanisms contains the full set of additive linear score functions with monotonically decreasing coefficients. Let $\pi_{NI\text{-}C}$ and $\pi_{NI\text{-}NC}$ be the total value provision of an optimal truthful additive linear score function for cases $NI\text{-}C$ and $NI\text{-}NC$, respectively (i.e., the optimal value of Equation 4.5 achieved by mechanisms characterized in Propositions 12 and 13, respectively). Then:

Theorem 5 $\pi_{NI\text{-}NC} \geq \pi_{NI\text{-}C}$.

The intuition behind Theorem 5 is that when a recipient knows about the other recipient's preferences, she may have an incentive to misreport (e.g., increase) the rankings of the products that are top-ranked by the other recipient (as explained after Proposition 12), while this incentive can be eliminated by withholding the competitor information. This result also has important implications because in practice, competitor information is not necessarily hidden from recipients. For example, in Beta, recipients can observe the set of products staged for shipment to other recipients during a warehouse tour at Beta; they may also learn about other recipients' preferences through previously shipped container mixes that are occasionally disclosed to recipients. In contrast, our result suggests that to be able to fully utilize recipients' revealed preferences in recipient selection decisions, MSROs should withhold the information regarding other recipients in the recipient pool.

Further, we emphasize through the following proposition that $\pi_{NI\text{-}C}$ and $\pi_{NI\text{-}NC}$ correspond to the smallest and largest value provision, respectively, among all belief structures regarding competitor information under the case without inventory information. In particular, let $\pi_{NI}(\phi_{1,2}, \phi_{2,1})$ denote the value provision of an optimal truthful additive linear score function for case NI where the two recipients' beliefs on each other's valuations are $\phi_{1,2}$ and $\phi_{2,1}$, respectively. Then:

Proposition 14 $\pi_{NI\text{-}C} \leq \pi_{NI}(\phi_{1,2}, \phi_{2,1}) \leq \pi_{NI\text{-}NC}, \forall \phi_{1,2}, \phi_{2,1}$.

Therefore, $\pi_{NI\text{-}C} - \pi_I$ and $\pi_{NI\text{-}NC} - \pi_I$ provide a lower and an upper bound, respectively, on the value of eliminating inventory information for a range of information sets regarding competitor information. Further, since $\pi_{NI\text{-}C}$ and $\pi_{NI\text{-}NC}$ are the optimal value provision for cases with and without competitor information, respectively, we use $\pi_{NI\text{-}NC} - \pi_{NI\text{-}C}$ to denote the value of eliminating competitor information. In §4.7, we quantify these values using real-life data.

The set of truthful additive linear score functions under different inventory and competitor information scenarios are summarized in Table 4.2. We remark that while we assume recipients' belief on inventory and competitor preference is perfectly symmetric in products in case NI-NC, we show in the appendix that under bounded perturbations on recipients' beliefs, recipients remain truthful as long as each recipient's valuations for different products are sufficiently different. We also note that under an additive linear score function, the coefficient g_j corresponds to each recipient's rank-j product instead of product j. This way, the announced g_j's do not reveal any information on inventory availability, and hence recipients will not be able to infer the MSRO inventory levels through the announced mechanism. We formalize this argument in the appendix.

4.6 Value of Cardinal Mechanisms

In previous sections, we have been focusing on ordinal mechanisms where recipients are only asked to report their preference rankings. In this section, we extend our analysis to consider cardinal mechanisms where recipients report their valuations for different prod-

Table 4.2: The set of truthful additive linear score functions under different information scenarios.

	With competitor information (C)	**Without competitor information (NC)**
With inventory information (I)	$g_1 = ... = g_N$	$g_1 = ... = g_N$
Without inventory information (NI)	$g_1 \geq g_2 = ... = g_N$ or $g_1 = ... = g_{N-1} \geq g_N$	$g_1 \geq ... \geq g_N$

ucts, and we investigate the value of cardinal mechanisms over ordinal mechanisms. Intuitively, a cardinal mechanism elicits more information than an ordinal mechanism, hence it is natural to expect that the optimal value provision of a cardinal mechanism should be higher than that of an ordinal mechanism. However, we show that for the class of implementable mechanisms we study, the extra information from eliciting exact valuations (beyond rankings) does not help improve the value provision.

We start with the definition of a cardinal mechanism, and we directly focus on the case without inventory information because when inventory information is provided to recipients, the incentive for recipients to misreport presented in Proposition 11 exists in the cardinal setting as well, and hence the only truthful mechanism is random selection among recipients. When inventory information is not provided to recipients, a cardinal mechanism is a function of the valuation profile $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ of the two recipients and the MSRO inventory levels $\boldsymbol{\mu}$, denoted by $p(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{\mu})$.

Let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ be the rankings that are consistent with valuations $\boldsymbol{v}_1, \boldsymbol{v}_2$, respectively. Define $p_1 \doteq p(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{\mu})$, $p_2 \doteq 1 - p(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{\mu})$, $x^0_{i,j}(\boldsymbol{v}_i, \boldsymbol{v}_{-i}, \boldsymbol{\mu}) \doteq p_i y^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})$, and $x^1_{i,j}(\boldsymbol{v}_i, \boldsymbol{v}_{-i}, \boldsymbol{\mu}) \doteq (1 - p_i) E_{\boldsymbol{\epsilon}}[y^1_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})]$. For each recipient i, given her true valuation $\boldsymbol{v}_i$, reported valuation $\boldsymbol{v}'_i$ and the inventory levels $\boldsymbol{\mu}$, her expected payoff at t is $\Pi^t_i(\boldsymbol{v}_i, \boldsymbol{v}'_i, \boldsymbol{\mu}) \doteq E_{\boldsymbol{v}_{-i}}[\sum_{j=1}^{N} v_{i,j} x^t_{i,j}(\boldsymbol{v}'_i, \boldsymbol{v}_{-i}, \boldsymbol{\mu})], t = 0, 1$. Then, the truthfulness of a cardinal mechanism can be defined in same manner as before.

As before, we consider mechanisms that are characterized by additive linear score functions, which now capture both ordinal and cardinal information. In particular, for

$i = 1, 2; j = 1, ..., N$, let $\tilde{v}_{i,j} = \frac{v_{i,j}}{\sum_{j'=1}^{N} v_{i,j'}} \in [0, 1)$ be the normalized valuation of recipient i for product j. Define:

$$s_i = \sum_{j=1}^{N} g_j \mu_{\rho_{i,j}} + \sum_{j=1}^{N} h(\tilde{v}_{i,j}) \mu_j, i = 1, 2, \tag{4.8}$$

where we assume that $h(0) = 0$ and $h(\tilde{v}_{i,j})$ is nondecreasing in $\tilde{v}_{i,j}$. We focus on normalized valuations because if recipients are prioritized based on their absolute valuations, they may have the incentive to present themselves as high valuation recipients (hence if h was a function of $v_{i,j}$, then we can trivially conclude that $h(v_{i,j})$ has to be zero for all $v_{i,j}$ to ensure truth telling). Further, by considering normalized valuations, we allow $h(\tilde{v}_{i,j})$ to depend on the valuations for all products: When a recipient's valuation of a product increases, she gets a higher score for each unit of that product in MSRO inventory but a lower score for all other products, which is very intuitive.

The score function for the cardinal setting (Equation 4.8) clearly includes that for the ordinal setting (Equation 4.7) as a special case. By eliciting more information, one may expect that a cardinal mechanism can do strictly better than an ordinal mechanism. However, as we show in the following proposition, the additional valuation information does not help improve the total value provision (while we only present the result for case NI-NC below, the same conclusion holds for case NI-C).

Proposition 15 *For case NI-NC, a score function $s_i = \sum_{j=1}^{N} g_j \mu_{\rho_{i,j}} + \sum_{j=1}^{N} h(\tilde{v}_{i,j}) \mu_j, i = 1, 2$ with $h(0) = 0$ and h nondecreasing is truthful if and only if $g_1 \geq ... \geq g_N$ and $h(\tilde{v}_{i,j}) = 0, \forall \tilde{v}_{i,j} \in [0, 1)$.*

Proposition 15 says that a truthful mechanism can only incorporate ordinal information. The underlying intuition is as follows: Under a cardinal mechanism in Equation 4.8, while both g_j's and the h function are chosen by the MSRO, recipients have the flexibility to manipulate the score they get for each unit of product j in the MSRO inventory by reporting different values of $v_{i,j}$ (or equivalently, $\tilde{v}_{i,j}$). In particular, suppose the h function is not a constant. Then, a recipient may have an incentive to misreport (increase) the valuations of her top-ranked products so as to gain a larger probability of being served should these products be available in the MSRO inventory.

Let $\pi_{NI\text{-}NC}$ and $\pi^{C}_{NI\text{-}NC}$ be the total value provision of an optimal additive linear score function for case $NI\text{-}NC$ in ordinal and cardinal settings, respectively. Then, we have the following result.

Theorem 6 $\pi^{C}_{NI\text{-}NC} = \pi_{NI\text{-}NC}$.

Theorem 6 implies that under a wide class of implementable mechanisms (i.e., additive linear score functions), there is no value added from eliciting exact valuations in addition to rankings. This finding differs from those in many existing studies on resource allocation in the literature, where the value of cardinal mechanisms is typically positive and could be arbitrarily large [e.g., 94, 95, 96]. However, as discussed earlier in §4.2, these studies either consider a large market or assume that cardinal information is given a priori (note that cardinal information will be valuable in our setting as well if it is given a priori). Another difference between the above studies and ours is that the above studies focus on determining what each player receives in a static setting, while in our setting recipients are served sequentially and our decision is to determine which recipient to serve at each shipping opportunity. In the former case, the outcome of a mechanism is a multi-dimensional allocation vector that specifies the probability for each recipient to receive each product. In our setting however, the outcome is a single-dimensional number that specifies the probability for each recipient to be selected, which limits the degree of freedom for leveraging the cardinal information. In that regard, our result parallels that in a two-product assignment problem, where the allocation vector for each recipient becomes single-dimensional (the two probabilities sum up to one), in which case it has been shown that there is no value added from eliciting cardinal information [128].

The implication of Theorem 6 is that it is sufficient for the MSRO to ask recipients to report their preference rankings under additive linear score function mechanisms. This is appealing from a practical perspective, because it is much easier for recipients to rank different products rather than to determine their valuations. Indeed, in other settings such as voting and school assignment, even though it has been shown that the value of cardinal information can be arbitrarily large, ordinal mechanisms are still widely considered both in the literature and in practice [**carroll2017mechanisms**].

4.7 Numerical Results

We now evaluate the performance of the proposed mechanisms and estimate the value of eliminating inventory and competitor information using Beta data.

Medical surplus products are typically classified into two broad categories: medical supplies and biomedical equipment. Medical supplies (e.g., masks, gloves, syringes, surgical instruments, etc.) usually have lower value and higher donation volume, while biomedical equipment (e.g., ultrasound units, examination tables, infant incubators, etc.) usually has higher value and lower donation volume. Our interactions with managers from Beta revealed that determining the fit between recipient needs and the available inventory of biomedical equipment is especially challenging and important to Beta's value provision because of the limited supply and the heterogeneous and unknown recipient preferences of these products. Hence, in our numerical analysis, we focus on recipient selection decisions determined by the availability of critical biomedical equipment.

Our data set contains biomedical equipment shipping data from 39 containers shipped between July 2015 and April 2016 at Beta. We develop a calibrated numerical study based on initial inventory levels, donation arrival rates and product valuations derived from this data set (please refer to the appendix for a detailed description). In particular, during the above period, Beta provided recipients access to its inventory database and allowed them to select their own products from the available inventory. This practice provides the basis for us to construct a set of recipient-specific product valuations that are consistent with recipients' observed picking behavior and allow for a systematic comparison of the value provision under different mechanisms.

4.7.1 Comparison of Value Provision under Different Mechanisms

Among the more than one hundred kinds of biomedical equipment Beta shipped out during the considered period of this study, we focus on the allocation of twenty so-called "Tier-1" biomedical equipment in Beta's practice. The Tier-1 equipment has high donor values and low supply volumes compared with other equipment, and hence is deemed the most critical and challenging to allocate. From the shipment data for the 39 containers considered in our study, we observe that on average each recipient receives about five pieces of the Tier-1 equipment, and typically at most one or two pieces within each equipment category due to the scarcity of these products (e.g., one ECG monitor, one infant incubator, one ultrasound unit and two electronic beds). In order to compare different mechanisms under a unified framework, we assume that each recipient receives $K = 5$ pieces of products among the considered $N = 20$ product categories and receives at most one piece within each product category. We consider the following three mechanisms:

- Mechanism I: the mechanism characterized in Proposition 11, where each recipient is selected with an equal probability; we also call this mechanism a random priority mechanism.

- Mechanism NI-C: a mechanism characterized in Proposition 12, where parameters for the score function are chosen as $g_1 = 1$, $g_2 = ... = g_N = 0$ (with the tie-breaking rule modified for the multi-recipient setting as described in the appendix: If two recipients have the same rank-1 product, then we randomly select a recipient and update her score as the inventory level of her rank-2 product, and so on). This mechanism is truthful when inventory information is not provided.

- Mechanism NI-NC: a mechanism characterized in Proposition 13, where parameters for the score function are chosen as $g_j = K - j + 1$ for $j \leq K$ and $g_j = 0$ for

$K < j \leq N$.[7] This mechanism is truthful when neither inventory nor competitor information is provided.

Note that mechanism I to some extent mimics the existing practice where the recipient selection decisions are not based on recipients' preferences. To characterize the value loss due to recipient needs being private information, we would also like to benchmark the proposed mechanisms with the first-best recipient selection sequence assuming that recipient needs are known to the MSRO. However, the optimal sequence for a multi-period problem is computationally intractable due to the dynamic and high-dimensional nature of our problem. Therefore, we instead compare with an optimal *certainty-equivalent* (CE) solution and a *clairvoyant* solution, a lower and upper bound respectively on the first best solution. The former is defined as an optimal recipient selection sequence assuming that the arrival quantity of each product in each period is equal to its mean arrival rate, while the latter is defined as an optimal recipient sequence assuming that the future arrival quantities are known in advance. We implement the optimal CE solution on a rolling horizon basis. That is, after serving the first recipient and observing realizations of new arrivals in the next period, we re-compute an optimal CE solution for the remaining recipients, and so on.

Finally, Beta usually has about 5-10 recipients in the system for whom funding has been secured for a container shipment. In this section, we consider a 5-recipient case, mainly because an optimal CE/clairvoyant solution becomes computationally challenging as the number of recipients increases (e.g., finding an optimal CE/clairvoyant solution for a 10-recipient problem requires comparing $10! = 3628800$ scenarios). In §4.7.2, we discuss a larger number of recipients and show that the value added from the proposed mechanisms increases with the number of recipients.

To generate a problem instance, we randomly select 5 recipients from the pool of 39 recipients and simulate the inventory arrival process using the estimated arrival rates of each product. The expected total value provision of each mechanism (estimated as the mean

[7] Given the MSRO's belief on recipients valuations, the optimal parameters g_j's can be found by maximizing the total value provision to recipients; here we choose a set of well-structured parameters for practicality.

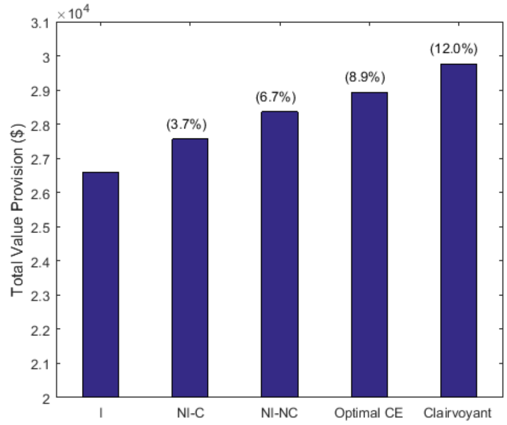


Figure 4.1: Total value provision of different mechanisms (percentages of improvement over the random priority mechanism are provided in parenthesis).

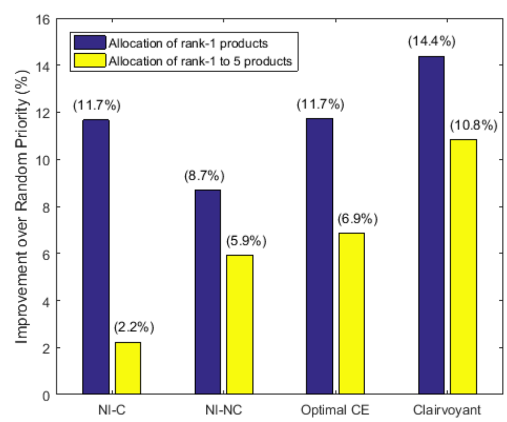


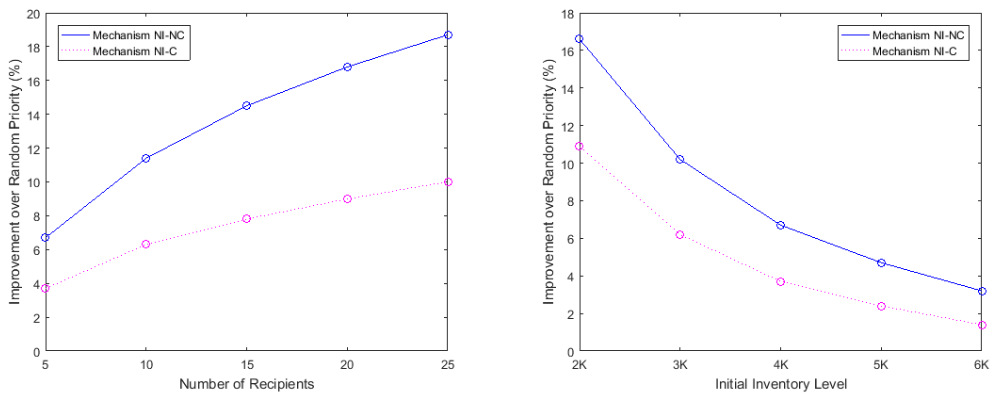
Figure 4.2: Performance improvement over the random priority mechanism regarding the allocation of rank-1 products and rank-1 to 5 products.

across 500 random problem instances) is presented in Figure 4.1. From these results, we observe that compared with the random priority mechanism I, all mechanisms that consider recipient needs information in recipient selection decisions significantly improve the total value provision. In particular, while the optimal CE solution and the clairvoyant solution improve the total value provision by 8.9% and 12.0%, respectively, mechanisms NI-C and NI-NC improve the total value provision by 3.7% and 6.7%, respectively. Hence, by eliminating both inventory and competitor information, we are able to close about 75% (≈6.7/8.9) of the gap between the optimal CE solution and the random priority mechanism, and 56% (≈6.7/12.0) of the gap between the clairvoyant solution and the random priority mechanism. We remark that such an improvement in value provision is significant, given that the sequence of donation arrivals and the set of recipients served are exactly the same under different mechanisms, and the only difference lies in when each recipient is served (and hence the container mix each recipient receives).

To better illustrate how the proposed mechanisms can help improve value provision to recipients, we further compare the proposed mechanisms with the random priority mechanism regarding the allocation of rank-1 products and rank-1 to 5 products (Figure 4.2). Clearly, both mechanisms NI-C and NI-NC significantly improve the allocation of rank-1 and rank-1 to 5 products (i.e., recipients receive more of their rank-1 and rank-1 to 5 products). In particular, mechanism NI-C, which prioritizes recipients whose rank-1 product is available, substantially improves the allocation of rank-1 products (11.7%), while also improving the allocation of rank-1 to 5 products (2.2%). On the other hand, mechanism NI-NC improves the allocation of rank-1 products (rank-1 to 5 products, respectively) by 8.7% (5.9%, respectively), which closes about 74% (86%, respectively) of the gap between the optimal CE solution and the random priority mechanism, and 60% (55%, respectively) of the gap between the clairvoyant solution and the random priority mechanism.

4.7.2 Effects of Recipient and Product Characteristics

Intuitively, both the recipient and product characteristics can potentially affect the value of the proposed mechanisms. We next analyze how the total number of recipients, total inventory levels of products, and heterogeneity among recipient needs affect our numerical results.



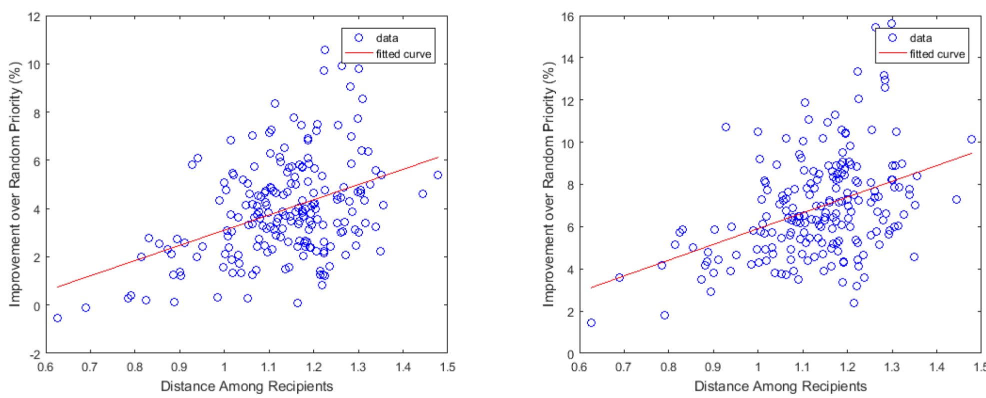
(a) (b)

Note: K in (b) represents the total amount of products each recipient receives and is equal to 5 in our numerical study. The initial inventory level in Figure (a) is equal to $4K$ while the number of recipients in Figure (b) is equal to 5.

Figure 4.3: Performance of mechanisms NI-C and NI-NC under different numbers of recipients and initial inventory levels.

First, we examine the effect of the number of recipients. In particular, we consider a range of 5 to 25 recipients. The value added from mechanisms NI-C and NI-NC (compared with random priority) under different number of recipients are presented in Figure 4.3 (a). From these results, we observe that the value of both mechanisms increases as the number of recipients increases. This is intuitive because when the number of recipients increases, it is more likely to find a recipient from the recipient pool whose needs are best met by the available inventory, and hence the benefit of appropriately selecting a recipient increases. The implication of this result is that the proposed mechanisms are particularly valuable for MSROs who serve large bases of recipients.

Second, we examine the effect of total inventory levels. In particular, we scale the inventory levels of all products proportionally to generate cases with different initial inventory levels (Figure 4.3 (b)). From these results, we observe that the value of the proposed mechanisms decreases as the inventory level increases. This is also intuitive because when the inventory levels are tight, the value provision may be small for a randomly selected recipient, and hence the benefit of appropriately selecting a recipient is large. When the inventory levels are high, this benefit becomes smaller. The implication of this result is that the proposed mechanisms are particularly valuable for MSROs who have a limited volume of donations (e.g., for critical products such as biomedical equipment) or maintain low inventory levels to keep warehousing costs low and inventory turnover high, practices MSRO aspire to because they signal operational excellence to potential donors.



(a) Mechanism $NI\text{-}C$ (b) Mechanism $NI\text{-}NC$

Figure 4.4: Performance of mechanisms $NI\text{-}C$ and $NI\text{-}NC$ under different distances among recipients.

Finally, we study the effect of heterogeneity among recipients. To do so, we first define a distance measure to quantify the heterogeneity among recipients' valuations. Consider two recipients with valuations $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively. Recall $\tilde{v}_{i,j} = v_{i,j}/\sum_{j=1}^{N} v_{i,j}$. Then, we define the distance between these two recipients as the L_1-norm of the difference between their normalized valuations, i.e., $\sum_{j=1}^{N} |\tilde{v}_{1,j} - \tilde{v}_{2,j}|$. For the case with more than two recipients, we define the distance among recipients as the average of the distance between each pair of recipients. Similar to the base scenario, we consider a 5-recipient case. The value added from mechanisms $NI\text{-}C$ and $NI\text{-}NC$ under different distances among recipients are presented in Figure 4.4, where each data point represents one problem instance generated by randomly selecting 5 different recipients from the recipient pool. From these results, we observe that the value added from the proposed mechanisms increases as the distance among recipients increases (the hypothesis that the slope is positive is supported at a p-value < 0.01 in both cases). The implication of this result is that the proposed mechanisms are particularly valuable for MSROs who serve heterogeneous recipients (e.g., recipients that are from different regions or those representing healthcare facilities with different specializations and needs).

4.8 Conclusion

The unique challenges faced by MSROs and the possible lack of operational expertise therein make matching supply and demand in an MSRO supply chain particularly difficult [85]. In particular, the key barrier that limits MSROs' value provision capabilities in this context is unknown recipient needs, which makes it challenging for the MSRO to determine the ideal recipient to serve at each shipping opportunity. Accordingly, we propose a mechanism design approach to elicit recipients' preference information to improve MSRO decision making.

We first show that what is considered industry best practice, i.e., providing recipients with MSRO inventory information to reduce the supply and demand mismatch, has its shortcomings. More specifically, we find that when inventory information is shared with recipients, the set of truthful mechanisms is highly restricted, which precludes using recipient-specific information in MSRO decision making. We then show that eliminating inventory information provision enlarges the set of truthful mechanisms, thereby increasing the total value provision to recipients. We also show that further withholding information about other recipients leads to an even larger set of truthful mechanisms and an additional increase in the total value provision. Finally, we show that it suffices for MSROs to elicit recipients' product ranking information under a class of implementable mechanisms, and there is no additional benefit from eliciting recipients' exact valuations. We also provide additional results that extend these insights to other scenarios (with multiple recipients, recipient-specific score functions, bounded perturbations on recipient beliefs, Bayesian update of recipient beliefs on MSRO inventory, and an alternative definition of truthfulness) in the appendix.

Based on these findings, we recommend that MSROs use a strategy consisting of a product ranking-based scoring mechanism in conjunction with withholding inventory and competitor information. Under this strategy, recipients are first asked to report their preference rankings over the full set of critical products the MSRO has access to (without providing inventory or competitor information). Then, recipients are prioritized based on a simple and intuitive scoring rule, and selected recipients are provided with the best bundle that meets their needs from the available inventory. Through a calibrated numerical study based on partner MSRO data, we show that this strategy can significantly improve MSROs' value provision capabilities. Moreover, the added value from this strategy increases in recipient heterogeneity and decreases in inventory levels, implying that this strategy is especially valuable for MSROs who serve large bases of heterogeneous recipients, or have highly constrained supplies of products and maintain low inventory levels.

We further note that our proposed strategy is flexible and can be adapted to handle other practical complexities. For example, when the number of product categories increases, it may be difficult to elicit recipients' full rankings of products. In this case, one could ask recipients to rate each product on a scale (with constraints on the number of products in each rating category). This yields a partial ranking of the products, which can be accommodated by appropriately defining the score functions (e.g., each recipient gets the same score for each unit of product within the same rating category). Our proposed strategy can also offer additional benefits beyond the scope of this paper. For example, eliciting recipient preferences may not only help MSROs better allocate their existing inventory, but also help improve the medical surplus acquisition processes (e.g., to solicit more donations of product categories that are highly ranked by many recipients).

As one of the first studies that model and analyze medical surplus recovery supply chains, we hope that this paper will encourage future work in this area. For example, in addition to determining the right recipient to serve at each shipping opportunity, another way of reducing the supply-demand mismatch is to wait until the MSRO inventory is a better match for the needs of recipients waiting to be served. A key trade-off here is that waiting imposes an opportunity cost on recipients. Hence, it would be interesting to determine container shipment times based on an evaluation of recipient needs relative to MSRO inventory availability. Another way of further improving the total value provision to recipients is to optimally determine the container mixes based on the reported preferences of all recipients. While we present a heuristic approach to improve upon the existing practice (i.e., the best bundle assumption) in the appendix, we remark that the general problem is challenging as even the centralized version of the problem (i.e., assuming recipient valuations are known to the MSRO) is intractable due to the high dimensionality of both state and action spaces. Finding competitive yet simple and implementable solutions that simultaneously determine which recipient to serve and which products to allocate to the chosen recipient is another interesting problem that we leave for future research.

Appendices

APPENDIX A

APPENDIX FOR CHAPTER 2

A.1 Proofs of Analytical Results

Proof of Lemma 1. From the system dynamics, we have $\sum_{k=1}^{K-1} X^{\pi}_{k,t+1} = (Y^{\pi}_t - D_t)^+ - (X^{\pi}_{K-1,t} - D_t)^+$, where $(Y^{\pi}_t - D_t)^+$ is the amount of inventory after demand realization at period t, and $(X^{\pi}_{K-1,t} - D_t)^+$ is the amount of outdates at period t. Then we have:

$$
\begin{aligned}
\hat{\mathscr{C}}(\pi) - \mathscr{C}(\pi) &= \sum_{t=1}^{T} \beta^{t-1}\hat{c}\Big(Q^{\pi}_t + (D_t - Y^{\pi}_t)^+ - (1-\beta)(Y^{\pi}_t - D_t)^+ - \beta(X^{\pi}_{K-1,t} - D_t)^+\Big) - \beta^T \hat{c} \sum_{k=1}^{K-1} X^{\pi}_{k,T+1} \\
&= \sum_{t=1}^{T} \beta^{t-1}\hat{c}\Big(Q^{\pi}_t + (D_t - Y^{\pi}_t)^+ - (Y^{\pi}_t - D_t)^+\Big) + \sum_{t=1}^{T} \beta^t \hat{c} \sum_{k=1}^{K-1} X^{\pi}_{k,t+1} - \beta^T \hat{c} \sum_{k=1}^{K-1} X^{\pi}_{k,T+1} \\
&= \sum_{t=1}^{T} \beta^{t-1}\hat{c}(D_t - \sum_{k=1}^{K-1} X^{\pi}_{k,t}) + \sum_{t=1}^{T-1} \beta^t \hat{c} \sum_{k=1}^{K-1} X^{\pi}_{k,t+1} \\
&= \sum_{t=1}^{T} \beta^{t-1}\hat{c}(D_t - \sum_{k=1}^{K-1} X^{\pi}_{k,t}) + \sum_{t=2}^{T} \beta^{t-1} \hat{c} \sum_{k=1}^{K-1} X^{\pi}_{k,t} \\
&= \sum_{t=1}^{T} \beta^{t-1}\hat{c} D_t,
\end{aligned}
$$

where the second equality follows from the fact that $\sum_{k=1}^{K-1} X^{\pi}_{k,t+1} = (Y^{\pi}_t - D_t)^+ - (X^{\pi}_{K-1,t} - D_t)^+$ as explained above, and the third equality comes from the fact that $(D_t - Y^{\pi}_t)^+ - (Y^{\pi}_t - D_t)^+ = D_t - Y^{\pi}_t$. □

Proof of Lemma 2. Without loss of generality, assume that $\tau_i - \tau_1 \leq K-1$ (otherwise, we can start from the largest $\tau_i - \tau_j$ that is less than or equal to $K-1$). By construction of policy IM, we have $\sum_{k=\tau_i-\tau_1}^{K-1} x^{IM}_{k,\tau_i} = \sum_{k=\tau_i-\tau_1}^{K-1} x^{B}_{k,\tau_i}$ and $x^{IM}_{k,\tau_i} = 0, k = \tau_i - \tau_1 + 1, ..., K-1$. Therefore, Inequality 2.3 holds for $k = \tau_i - \tau_1, ..., K-1$.

Then, for $j = 2, ..., i-1$, by construction of policy IM, we have $\sum_{k=\tau_i-\tau_{j-1}}^{K-1} x^{IM}_{k,\tau_i} = \sum_{k=\tau_i-\tau_{j-1}}^{K-1} x^{B}_{k,\tau_i}$, $\sum_{k=\tau_i-\tau_j}^{K-1} x^{IM}_{k,\tau_i} = \sum_{k=\tau_i-\tau_j}^{K-1} x^{B}_{k,\tau_i}$ and $x^{IM}_{k,\tau_i} = 0, k = \tau_i - \tau_j + 1, ..., \tau_i - \tau_{j-1} - 1$. Therefore, Inequality

2.3 holds for $k = \tau_i - \tau_j, ..., \tau_i - \tau_{j-1} - 1$.

Finally, by construction of policy IM, we have $\sum_{k=\tau_i-\tau_{i-1}}^{K-1} x_{k,\tau_i}^{IM} = \sum_{k=\tau_i-\tau_{i-1}}^{K-1} x_{k,\tau_i}^{B}$ and $x_{k,\tau_i}^{IM} = 0, k = 1, ..., \tau_i - \tau_{i-1} - 1$. Therefore, Inequality 2.3 holds for $k = 1, ..., \tau_i - \tau_{i-1} - 1$, which completes the proof.

Proof of Lemma 3. We prove this lemma in the following two steps:

Step 1: We first prove statement (iii) holds for all $t = 1, ..., T$ under Assumption 1, i.e., for all $t = 1, ..., T$, $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1$, $\forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$. Suppose at $t+1$, there exist some $\mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$ and some $f_{t+1} \in \mathscr{F}_{t+1}$ such that $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) > w/\beta$ for some $k \in \{1, ..., K-1\}$. We next show that FIFO is not an optimal issuing policy for some inventory vector and demand at period t (i.e., Assumption 1 does not hold). In particular, consider any demand $d_t > 0$, and let $\mathbf{x}_t$ and q_t be such that $x_{k-1,t} = x_{k,t+1} + \epsilon, x_{m-1,t} = x_{m,t+1}, m = 1, ..., k-1, k+1, ..., K-1$, and $x_{K-1,t} = d_t$, where $x_{0,t} = q_t$ and ϵ is positive but sufficiently small such that $\epsilon \leq d_t$ and $\sum_{k=1}^{K-1} x_{k,t} + q_t = \sum_{k=1}^{K-1} x_{k,t+1} + d_t + \epsilon \leq \bar{y}_t$. Then, FIFO issuing policy will issue d_t units of age $K-1$. Consider another issuing policy γ which issues $d_t - \epsilon$ units of age $K-1$ and ϵ units of age $k-1$. Then, there will be ϵ more units of outdates under issuing policy γ and ϵ more inventory of age k at the beginning of period $t+1$ under FIFO issuing policy. By assumption, $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) > w/\beta$; thus γ is strictly better than FIFO, which is a contradiction.

Step 2: We next prove that if statement (iii) holds for all $t = 1, ..., T$, then statements (i) and (ii) both holds for all $t = 1, ..., T$. Statements (i) and (ii) clearly both hold for T since $C_{T+1}(\mathbf{x}_{T+1}, f_{T+1}) = 0, \forall \mathbf{x}_{T+1}, f_{T+1}$. Assume that Statements (i) and (ii) hold for t. We next show that they also hold for $t-1$.

We first show $C_t^{(k)}(\mathbf{x}_t, f_t) \geq 0, k = 1, ..., K-1, \forall \mathbf{x}_t, f_t$. Consider the following two cases.

Case 1: Suppose $\sum_{k=1}^{K-1} x_{k,t} < \bar{y}_t$. Consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}_t$ and System 2 starts from $\mathbf{x}'_t$, where $x'_{k,t} = x_{k,t} + \epsilon$ and $x'_{m,t} = x_{m,t}, m = 1, ..., k-1, k+1, ..., K-1$ (i.e., System 2 has ϵ more units of age k), and ϵ is positive but sufficiently small such that $\sum_{k=1}^{K-1} x_{k,t} + \epsilon \leq \bar{y}_t$. Let System 2 follow an optimal ordering decision rule at each period, and let System 1 order ϵ more units than System 2 at t and follow an optimal ordering decision rule afterward. Then, it is sufficient to show that System 1 has no more expected total cost than System 2. Clearly, both the shortage penalty and the holding cost at period t in the two systems are the same. Also, the total inventory level after ordering in both systems must be no more than $\bar{y}_t$, because ordering up to more than $\bar{y}_t(\geq \tilde{y}_t(f_t))$ increases both the expected cost at period t and the future optimal cost-to-go (since $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \geq 0, k = 1, ..., K-1, \forall \mathbf{x}_{t+1}, f_{t+1}$ by induction assumption). For any realization of demand d_t, let $\mathbf{x}_{t+1}$ and $\mathbf{x}'_{t+1}$ be the inventory vectors at period $t+1$ for Systems 1 and 2, respectively. Assume that there are $\xi \leq \epsilon$ more units of outdates in System 2 than in System 1 at period t. Then $\sum_{k=1}^{K-1} x_{k,t+1} = \sum_{k=1}^{K-1} x'_{k,t+1} + \xi \leq \bar{y}_t$ and $\sum_{k=m}^{K-1} x_{k,t+1} \leq \sum_{k=m}^{K-1} x'_{k,t+1}, m = 2, ..., K-1$. By induction assumption, we have $C_{t+1}^{(i)}(\mathbf{x}_{t+1}, f_{t+1}) \leq C_{t+1}^{(j)}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, 1 \leq i < j \leq K-1, \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t, \forall f_{t+1}$. Therefore, System 1 has no more expected total cost than System 2.

Case 2: Suppose $\sum_{k=1}^{K-1} x_{k,t} \geq \bar{y}_t$. Consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}_t$ and System 2 starts from $\mathbf{x}'_t$, where $x'_{k,t} = x_{k,t} + \epsilon$ and $x'_{m,t} = x_{m,t}, m = 1, ..., k-1, k+1, ..., K-1$ (i.e., System 2 has ϵ more units of age k), and ϵ is any positive number. Let System 2 follow an optimal ordering decision rule at each period, and let System 1 order nothing at period t and follow an optimal ordering decision rule afterward. Then, it is sufficient to show that System 1 has no more expected total cost than System 2. Let y_t and y'_t be the total inventory levels after ordering at period t in Systems 1 and 2, respectively. Then, $\bar{y}_t \leq y_t \leq y'_t$. Thus the expected cost at period t in System 1 is no more than that in System 2 (because a total inventory level of $\tilde{y}_t(f_t)(\leq \bar{y}_t)$ minimizes the expected sum of shortage penalty and holding cost at period t and the costs are convex in order quantities). For any demand realization d_t, let $\mathbf{x}_{t+1}$ and $\mathbf{x}'_{t+1}$ be the inventory vectors at period $t+1$ for Systems 1 and 2, respectively. Then we have $x_{k,t+1} \leq x'_{k,t+1}, k = 1, ..., K-1$. By induction assumption, we have $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \geq 0, k = 1, ..., K-1, \forall \mathbf{x}_{t+1}, f_{t+1}$. Therefore, System 1 has no more expected total cost than System 2.

Combining both Cases 1 and 2, we have $C_t^{(k)}(\mathbf{x}_t, f_t) \geq 0, k = 1, ..., K-1, \forall \mathbf{x}_t, f_t$.

We next show $C_t^{(i)}(\mathbf{x}_t, f_t) \leq C_t^{(j)}(\mathbf{x}_t, f_t), 1 \leq i < j \leq K-1, \forall \mathbf{x}_t$ such that $\sum_{k=1}^{K-1} x_{k,t} < \bar{y}_{t-1}$, $\forall f_t$. Given $\mathbf{x}_t, f_t$ such that $\sum_{k=1}^{K-1} x_{k,t} < \bar{y}_{t-1}$, consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}'_t$ and System 2 starts from $\mathbf{x}''_t$, where $x'_{i,t} = x_{i,t} +$

$\epsilon, x'_{k,t} = x_{k,t}, k \neq i$, and $x''_{j,t} = x_{j,t} + \epsilon, x''_{k,t} = x_{k,t}, k \neq j, 1 \leq i < j \leq K-1$ (i.e., System 1 starts with ϵ more units of age i and System 2 with ϵ more units of age j), and ϵ is positive but sufficiently small such that $\sum_{k=1}^{K-1} x_{k,t} + \epsilon \leq \bar{y}_t$. Let System 2 follow an optimal ordering decision rule at each period, and let System 1 order the same amount as System 2 at t and follow an optimal decision rule afterward. Then, it is sufficient to show that System 1 has no more expected total cost than System 2. For any demand realization d_t, let $\mathbf{x}'_{t+1}$ and $\mathbf{x}''_{t+1}$ be the inventory vectors at period $t+1$ in Systems 1 and 2, respectively. Then, we have $x'_{k,t+1} = x''_{k,t+1}, k = 1, ..., i, x'_{i+1,t+1} \geq x''_{i+1,t+1}$, and $x'_{k,t+1} \leq x''_{k,t+1}, k = i+2, ..., K-1$. Assume that there are $\xi \leq \epsilon$ more units of outdates in System 2 than in System 1 at period t. Then $\sum_{k=1}^{K-1} x'_{k,t+1} = \sum_{k=1}^{K-1} x''_{k,t_0+1} + \xi \leq \bar{y}_t$. By induction assumption, we have $C^{(i)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq C^{(j)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, 1 \leq i < j \leq K-1, \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t, \forall f_{t+1}$. Therefore, System 1 has no more expected total cost than System 2, which completes the proof. $\square$

Proof of Lemma 4. As discussed in the proof of Lemma 3, if the initial inventory level at period t is less than or equal to $\bar{y}_t$, then the total inventory level after ordering under an optimal ordering decision rule must also be less than or equal to $\bar{y}_t$ (because ordering up to more than $\bar{y}_t$ increases both the expected cost at period t and the future optimal cost-to-go). Therefore, given that we start from zero inventory and an optimal ordering decision rule is followed at each period under policy IM, we have $\sum_{k=1}^{K-1} x^{IM}_{k,t+1} \leq \bar{y}_t, t = 1, ..., T$. Similarly, for the case where we start from a high inventory level, the ordering quantity under policy IM will always be zero until period t such that $\sum_{k=1}^{K-1} x^{IM}_{k,t} \leq \bar{y}_t$, before which no units will be moved under policy IM since the inventory level under policy IM will be no more than that under policy B. Therefore, we always have $\sum_{k=1}^{K-1} x^{IM}_{k,t+1} \leq \bar{y}_t$ at $t+1$ if units are moved at t.

Recall that for each given sample path, $\mathscr{T}_H = \{\tau_1, ..., \tau_n\}$. Consider a variation of policy IM, call it IM_1; under IM_1, the movements of units are only performed at τ_1, and an optimal ordering rule is followed and no movements are performed at the following periods. Then, to show $\mathrm{E}[\mathscr{C}(IM)] \leq \mathrm{E}[\mathscr{C}(OPT)]$, it is sufficient to show $\mathrm{E}[\mathscr{C}(IM_1)] \leq \mathrm{E}[\mathscr{C}(OPT)]$; since if this is true, following a similar argument, movements at future periods can only further decrease the expected total cost. Consider any realization of τ_1. Clearly, the total cost under policies IM_1 and OPT are the same for all periods $1, ..., \tau_1 - 1$. Without loss of generality, further assume that at τ_1, we only moved ϵ units of age k to age zero, $k = 1, ..., K-1$. Then, after the movements, there are ϵ more units of age zero but ϵ fewer units of age k under policy IM_1 than under policy OPT.

Consider the following two cases. First, suppose the amount of outdates at τ_1 under policies IM_1 and OPT are the same. Then, the total cost at τ_1 under the two policies are the same, and total inventory level at $\tau_1 + 1$ under the two policies are also the same but the inventory vector under policy IM_1 is "younger", i.e., $\sum_{k=1}^{K-1} x^{IM_1}_{k,\tau_1+1} = \sum_{k=1}^{K-1} x^{OPT}_{k,\tau_1+1}$, and $\sum_{k=m}^{K-1} x^{IM_1}_{k,\tau_1+1} \leq \sum_{k=m}^{K-1} x^{OPT}_{k,\tau_1+1}, m = 2, ..., K-1$. By Lemma 3, we have $C^{(i)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq C^{(j)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}), 1 \leq i < j \leq K-1, \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t, \forall f_{t+1}$. Therefore, policy IM_1 has no more expected total cost than policy OPT.

Second, suppose there are $\xi \leq \epsilon$ more units of outdates at τ_1 under policy OPT than under policy IM_1 (this is only possible when we moved units of age $K-1$ to age zero under policy IM_1). Then, we have $\sum_{k=1}^{K-1} x^{IM_1}_{k,\tau_1+1} = \sum_{k=1}^{K-1} x^{OPT}_{k,\tau_1+1} + \xi$, and $\sum_{k=m}^{K-1} x^{IM_1}_{k,\tau_1+1} \leq \sum_{k=m}^{K-1} x^{OPT}_{k,\tau_1+1}, m = 2, ..., K-1$. By Lemma 3, we have $C^{(i)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq C^{(j)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, 1 \leq i < j \leq K-1, \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t, \forall f_{t+1}$. Therefore, policy IM_1 has no more expected cost than policy OPT, which completes the proof. $\square$

Proof of Proposition 1. We start with proving a structural property on the optimal cost-to-go function under the marginal-cost accounting scheme. For $t = 1, ..., T$, given $\mathbf{x}_t$ and f_t, let $\hat{C}_t(\mathbf{x}_t, f_t)$ denote the optimal cost-to-go function at period t under the marginal-cost accounting scheme, and let $\Gamma_t(\mathbf{x}_t, f_t, q_t) = P_t(\mathbf{x}_t, f_t, q_t) + H_t(\mathbf{x}_t, f_t, q_t) + W_t(\mathbf{x}_t, f_t, q_t)$. Then, the optimality equation under the marginal-cost accounting scheme is as follows:

$$\hat{C}_t(\mathbf{x}_t, f_t) = \min_{q_t \geq 0} \left\{ \Gamma_t(\mathbf{x}_t, f_t, q_t) + \mathrm{E}[\hat{C}_{t+1}(\mathbf{X}_{t+1}, F_{t+1}) | f_t] \right\}.$$

For $k = 1, ..., K-1$, for the continuous case, let $\hat{C}^{(k)}_t(\mathbf{x}_t, f_t)$ denote the right partial derivative of $\hat{C}_t(\mathbf{x}_t, f_t)$ with respect to to $x_{k,t}$; for the discrete case, let $\hat{C}^{(k)}_t(\mathbf{x}_t, f_t)$ denote the incremental of $\hat{C}_t(\mathbf{x}_t, f_t)$ caused by a unit increase of $x_{k,t}$. Then, we have the following result.

Lemma 8 *Under Assumption 1, for* $t = 1, ..., T$, $\hat{C}^{(k)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq 0, k = 1, ..., K-1, \forall \mathbf{x}_{t+1}, f_{t+1}$.

Proof. The claim in Lemma 8 is clearly true for T since $\hat{C}_{T+1}(\mathbf{x}_{T+1}, f_{T+1}) = 0, \forall \mathbf{x}_{T+1}, f_{T+1}$.

Assume that the claim is true for t. We now show that it is also true for $t-1$. Consider the following two cases:

Case 1: Suppose $\sum_{k=1}^{K-1} x_{k,t} < \bar{y}_t$. Consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}_t$ and System 2 starts from $\mathbf{x}'_t$, where $x'_{k,t} = x_{k,t} + \epsilon$ and $x'_{m,t} = x_{m,t}, m = 1, ..., k-1, k+1, ..., K-1$ (i.e., System 2 has ϵ more units of age k), and ϵ is positive but sufficiently small such that $\sum_{k=1}^{K-1} x_{k,t} + \epsilon \le \bar{y}_t$. Let System 1 follow an optimal ordering decision rule at each period. To define the ordering policy in System 2, along a given sample path, let $t_0 \in (t, t+K-1]$ be such that at all periods $t, ..., t_0 - 1$, there are still some products that are ordered in periods $< t$ in System 2, while at the beginning of period t_0, all of these products are gone (either used to satisfy demand or outdated) and all products in inventory are ordered at periods $\ge t$. Then, for each period $t, ..., t_0 - 1$, let System 2 order up to the same level as System 1 (order nothing if System 2 already has more inventory than System 1), and let System 2 follow an optimal ordering decision rule in periods $\ge t_0$.

Then, to prove the lemma, it is sufficient to show that the expected total cost under the marginal-cost accounting scheme in System 2 is no more than that in System 1. By definition of t_0, no units ordered at periods $\ge t$ will be outdated by the beginning of period t_0. Then, the total cost under the marginal-cost accounting scheme in each system is comprised of the following three parts: i) the shortage penalties that occur at periods $t, ..., t_0 - 1$, ii) the holding costs that occur at periods $t, ..., t_0 - 1$ charged for units ordered at periods $\ge t$, and iii) the total costs (shortage penalties, holding and outdating costs) that occur at periods $\ge t_0$.

i) Consider the shortage penalties that occur at periods $t, ..., t_0 - 1$. By definition of the ordering policy under System 2, after ordering, there is at least the same amount of inventory in System 2 as that in System 1 at each period $t, ..., t_0 - 1$. Therefore, the total shortage penalty at periods $t, ..., t_0 - 1$ in System 2 is no more than that in System 1 with probability one.

ii) Consider the holding costs that occur at periods $t, ..., t_0 - 1$ charged for units ordered at periods $\ge t$. If the ordering quantity in System 2 is zero for all $t, ..., t_0 - 1$, then there is nothing to prove. Otherwise, let $s_0 \in [t, t_0)$ be the first period such that the ordering quantity in System 2 is strictly positive. Let q_s and q'_s be the ordering quantity in Systems 1 and 2 at period s, respectively, and y_s and y'_s be the total inventory level after ordering in Systems 1 and 2 at period s, respectively.

Since System 2 started with more inventory than System 1, the amount of outdates in System 2 is at least as much as that in System 1 at each period $t, ..., t_0 - 1$. Therefore, we have $\sum_{k=1}^{K-1} x'_{k,s} \leq \sum_{k=1}^{K-1} x_{k,s}, q'_s \geq q_s$, and $y'_s = y_s, \forall s = s_0 + 1, ..., t_0 - 1$. Units ordered at period t will be of age $t_0 - 1 - t$ at period $t_0 - 1$. Since System 2 started with more inventory than System 1, we must have $\sum_{k=t_0-t}^{K-1} x'_{k,t_0-1} \geq \sum_{k=t_0-t}^{K-1} x_{k,t_0-1}$. Therefore, we have $\sum_{k=1}^{t_0-t-1} x'_{k,t_0-1} \leq \sum_{k=1}^{t_0-t-1} x_{k,t_0-1}$. By definition of t_0, all units ordered at periods $\geq t$ in System 2 are still in inventory at the beginning of period $t_0 - 1$. Further considering $q'_s = 0, \forall s = t, ..., s_0 - 1$ and $q'_s \geq q_s, \forall s = s_0 + 1, ..., t_0 - 2$, the holding cost that occurs at periods $t, ..., t_0 - 2$ charged for units ordered at periods $\geq t$ in System 2 is no more than that in System 1 with probability one. It remains to consider the holding costs that occur at period $t_0 - 1$. At the beginning of period t_0, all units in inventory in both systems are ordered at periods $\geq t$, and there is no more inventory in System 2 than in System 1. Therefore, the holding cost that occurs at period $t_0 - 1$ charged for units ordered at periods $\geq t$ in System 2 is no more than that in System 1 with probability one.

iii) Consider the total costs that occur at periods $\geq t_0$. If the ordering quantity in System 2 is zero for all $t, ..., t_0 - 1$, then System 2 will be empty at the beginning of period t_0, and the expected total cost that occurs at periods $\geq t_0$ in System 2 is no more than that in System 1 because of $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \geq 0, k = 1, ..., K - 1, \forall \mathbf{x}_{t+1}, f_{t+1}$ (Lemma 3). Otherwise, define s_0 in the same way as in item ii). Following the same argument as in item ii), we have $\sum_{k=1}^{K-1} x'_{k,t_0} \leq \sum_{k=1}^{K-1} x_{k,t_0}$. Further considering $q'_s = 0, \forall s = t, ..., s_0 - 1$ and $q'_s \geq q_s, \forall s = s_0 + 1, ..., t_0 - 2$, we must have $\sum_{m=k}^{K-1} x'_{m,t_0} \leq \sum_{m=k}^{K-1} x_{m,t_0}, k = 1, ..., K - 1$. By Lemma 3, we have $0 \leq C_{t+1}^{(i)}(\mathbf{x}_{t+1}, f_{t+1}) \leq C_{t+1}^{(j)}(\mathbf{x}_{t+1}, f_{t+1}), 1 \leq i < j \leq K - 1, \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t, \forall f_{t+1}$. Therefore, the expected total cost that occurs at periods $\geq t_0$ in System 2 is no more than that in System 1.

Case 2: Suppose $\sum_{k=1}^{K-1} x_{k,t} \geq \bar{y}_t$. Consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}_t$ and System 2 starts from $\mathbf{x}'_t$, where $x'_{k,t} = x_{k,t} + \epsilon$ and $x'_{m,t} = x_{m,t}, m = 1, ..., k - 1, k + 1, ..., K - 1$ (i.e., System 2 has ϵ more units of age k), and ϵ is any positive number. Let both Systems 1 and 2 follow an optimal ordering decision rule at each period. Since $\sum_{k=1}^{K-1} x_{k,t} \geq \bar{y}_t$, the ordering quantities in both systems are zero at period t. Let y_t and y'_t be the total inventory levels after ordering in Systems 1 and 2, respectively. Then,

$\bar{y}_t \leq y_t \leq y'_t$. Hence the expected marginal shortage penalty at period t in System 2 is no more than that in System 1, and there is no marginal holding or outdating cost in either system. Let $\mathbf{x}_{t+1}$ and $\mathbf{x}'_{t+1}$ be the inventory vectors at period $t+1$ for Systems 1 and 2, respectively. Then we have $x_{k,t+1} \leq x'_{k,t+1}, k = 1, ..., K-1$. By induction assumption, we have $\hat{C}^{(k)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq 0, k = 1, ..., K-1, \forall \mathbf{x}_{t+1}, f_{t+1}$. Therefore, under the marginal-cost accounting scheme, System 2 has no more expected total cost than System 1. $\square$

With the above result, we next prove the proposition by contradiction. Suppose for some period t, given $\mathbf{x}_t$ and f_t, we have $q_t^L > q_t^{OPT}$. Consider a policy L, under which q_t^L units are ordered at period t and an optimal ordering decision rule is followed at the following periods. Then, the total expected cost of policy L is $\Gamma_t(\mathbf{x}_t, f_t, q_t^L) + \mathrm{E}[\hat{C}_{t+1}(\mathbf{X}^L_{t+1}, F_{t+1})|f_t]$. On the other hand, the total expected cost of policy OPT is $\Gamma_t(\mathbf{x}_t, f_t, q_t^{OPT}) + \mathrm{E}[\hat{C}_{t+1}(\mathbf{X}^{OPT}_{t+1}, F_{t+1})|f_t]$. By definition of q_t^L, we have $\Gamma_t(\mathbf{x}_t, f_t, q_t^L) < \Gamma_t(\mathbf{x}_t, f_t, q_t^{OPT})$. Further, since $q_t^L > q_t^{OPT}$, we have $X^L_{k,t+1} \geq X^{OPT}_{k,t+1}, k = 1, ..., K-1$ for any realization of D_t. Therefore, by Lemma 8, we have $\hat{C}_{t+1}(\mathbf{X}^L_{t+1}, F_{t+1}) \leq \hat{C}_{t+1}(\mathbf{X}^{OPT}_{t+1}, F_{t+1})$ with probability one. Then:

$$\Gamma_t(\mathbf{x}_t, f_t, q_t^L) + \mathrm{E}[\hat{C}_{t+1}(\mathbf{X}^L_{t+1}, F_{t+1})|f_t] < \Gamma_t(\mathbf{x}_t, f_t, q_t^{OPT}) + \mathrm{E}[\hat{C}_{t+1}(\mathbf{X}^{OPT}_{t+1}, F_{t+1})|f_t].$$

That is, policy OPT is not optimal for periods $t, ..., T$, which is a contradiction. $\square$

Proof of Theorem 2. The proof of Theorem 2 follows a similar way as that for Theorem 1. The main difference lies in the construction of the bridging policy, policy IM. In particular, now policy IM is constructed as follows: At each period t, given $\mathbf{x}_t$ and f_t, let the system under policy IM follow an optimal ordering decision rule. What differentiates policies IM and OPT is that under policy IM, at each period after ordering and before demand realization, 1) products in the inventory vector can be "moved" from older positions to the position of age 0; and 2) products of age 0 can be intendedly disposed.

At each period t, let y_t^{TB} and y_t^{IM} be the total inventory levels after ordering under policies TB and IM, respectively (before any disposal of units under policy IM at period t). Also, given $\mathbf{x}_t^{TB}$ and f_t, let y_t^B denote the total inventory level after ordering if the balancing ordering quantity q_t^B is ordered. Then, we assign period t into one of the following four subsets of the decision epochs:

$$\mathscr{T}_P = \{t : y_t^B \geq y_t^{IM}\}, \mathscr{T}_H = \{t : y_t^B < y_t^{IM}, y_t^{TB} = y_t^B\},$$

$$\mathscr{T}_{LH} = \{t : y_t^B < y_t^{IM}, y_t^{TB} > y_t^B\}, \mathscr{T}_{UH} = \{t : y_t^B < y_t^{IM}, y_t^{TB} < y_t^B\}.$$

Note that our partition of the decision epochs is different from that in the nonperishable case in [136] (in which case only two subsets are needed because of the optimality of a base-stock policy). We next define the rules of movements and disposals under policy IM in order to bound the the total shortage penalty of policy TB at each period $t \in \mathscr{T}_P \cup \mathscr{T}_{UH}$ and the total holding and outdating cost of policy TB charged for the first q_t^B units ordered at each period $t \in \mathscr{T}_H \cup \mathscr{T}_{LH}$. We allow movements of units at period t if $t \in \mathscr{T}_H \cup \mathscr{T}_{LH} \cup \mathscr{T}_{UH}$. Let $t \in \mathscr{T}_H \cup \mathscr{T}_{LH} \cup \mathscr{T}_{UH} = \{\tau_1, ..., \tau_n\}$. The rules of movements are defined in a similar way as before such that after the movements at each τ_i, we have:

(i) There is only positive inventory of age 0 and $\tau_i - \tau_j$ under policy IM, for all $j = 1, ..., i-1$ such that $\tau_j \in \mathscr{T}_H \cup \mathscr{T}_{LH}$.

(ii) For $j = 1, ..., i-1$ such that $\tau_j \in \mathscr{T}_H \cup \mathscr{T}_{LH}$, $\sum\limits_{k=\tau_i-\tau_j}^{K-1} x_{k,\tau_i}^{IM} = x_{\tau_i-\tau_j,\tau_i}^{B} + \sum\limits_{k=\tau_i-\tau_j+1}^{K-1} x_{k,\tau_i}^{TB}$, where $x_{\tau_i-\tau_j,\tau_i}^{B}$ denotes the inventory of age $\tau_i - \tau_j$ at period τ_i under policy TB if $q_{\tau_j}^B$ instead of $q_{\tau_j}^{TB}$ units are ordered at τ_j (note that this is achievable because we have $y_{\tau_j}^B < y_{\tau_j}^{IM}$ for all $\tau_j \in \mathscr{T}_H \cup \mathscr{T}_{LH}$; also note that property (ii) is equivalent to Equation (2.2) for $\tau_j \in \mathscr{T}_H$ because we have $q_{\tau_j}^{TB} = q_{\tau_j}^B$ for all $\tau_j \in \mathscr{T}_H$).

In addition to movements, we also allow disposals of units at period t if $t \in \mathscr{T}_{UH}$. For $t \in \mathscr{T}_{UH}$, we have $y_t^{TB} < y_t^B < y_t^{IM}$. After the movements of units, there must be at least $y_t^{IM} - y_t^{TB}$ units of age 0 under policy IM. Then, we dispose $y_t^{IM} - y_t^{TB}$ units of age 0 under policy IM so that after the disposal, we have $y_t^{IM} = y_t^{TB}$, and none of the above two properties resulted from movements of units is violated.

Then, similar as before, to show $\mathrm{E}[\mathscr{C}(TB)] \leq 2\mathrm{E}[\mathscr{C}(OPT)]$, it is sufficient to show $\mathrm{E}[\mathscr{C}(IM)] \leq \mathrm{E}[\mathscr{C}(OPT)]$ and $\mathrm{E}[\mathscr{C}(TB)] \leq 2\mathrm{E}[\mathscr{C}(IM)]$, respectively. We have shown in Lemma 4 that moving units from older to younger positions does not increase the expected total cost. We next show that disposing units at periods in $\mathscr{T}_{UH}$ does not increase the expected total cost either. For $t \in \mathscr{T}_{UH}$, since $y_t^{TB} < y_t^B$, by definition of policy TB, $y_t^{TB} = \sum_{k=1}^{K-1} x_{k,t}^{TB} + q_t^U$ provides an upper bound on the optimal *order-up-to level* (i.e., total inventory level after ordering) for given $\mathbf{x}_t^{TB}$ and f_t. Also, similar to before, the inventory vector under policy IM is "younger" than that under policy TB after the movements (i.e., $\sum_{m=k}^{K-1} x_{m,t}^{IM} \leq \sum_{m=k}^{K-1} x_{m,t}^{TB}, k = 1, ..., K-1$). Then, it is not difficult to show that y_t^{TB} also provides an upper bound on the optimal order-up-to level for given $\mathbf{x}_t^{IM}$ and f_t. Therefore, disposal of inventory from y_t^{IM} to y_t^{TB} will not increase the expected total cost. Then we have:

$$\mathrm{E}[\mathscr{C}(IM)] \leq \mathrm{E}[\mathscr{C}(OPT)]. \tag{A.1}$$

We next show $\mathrm{E}[\mathscr{C}(TB)] \leq 2\mathrm{E}[\mathscr{C}(IM)]$, which together with Inequality (A.1) lead to our conclusion. Let P_t^{TB}, H_t^{TB} and W_t^{TB} be the marginal shortage penalty, holding and outdating costs at period t under policy TB. Also, we slightly abuse the notation to let P_t^B, H_t^B and W_t^B denote the marginal shortage penalty, holding and outdating costs by following the balancing quantity q_t^B at period t, given $\mathbf{x}_t^{TB}$ and f_t. By construction of policy IM, after the movements and disposals, we have $y_t^B \geq y_t^{IM}, \forall t \in \mathscr{T}_P \cup \mathscr{T}_{UH}$. Then:

$$\sum_{t \in \mathscr{T}_P \cup \mathscr{T}_{UH}} P_t^B \leq \sum_{t=1}^{T} P_t^{IM}. \tag{A.2}$$

Define the dynamic unit-matching scheme in a similar way as before, such that the first q_t^B units ordered at each $t \in \mathscr{T}_H \cup \mathscr{T}_{LH}$ under policy TB are matched to units under policy IM on a one to one correspondence, and a matched unit under policy TB stays in inventory no longer than its matched unit under policy IM (recall that we have $\sum_{k=\tau_i-\tau_j}^{K-1} x_{k,\tau_i}^{IM} = x_{\tau_i-\tau_j,\tau_i}^B + \sum_{k=\tau_i-\tau_j+1}^{K-1} x_{k,\tau_i}^{TB}$ after the movements at period τ_i). Then:

$$\sum_{t \in \mathscr{T}_H \cup \mathscr{T}_{LH}} H_t^B \leq \sum_{t=1}^{T} H_t^{IM}, \quad \sum_{t \in \mathscr{T}_H \cup \mathscr{T}_{LH}} W_t^B \leq \sum_{t=1}^{T} W_t^{IM}. \tag{A.3}$$

Finally, recall that $\Gamma_t(\mathbf{x}_t, f_t, q_t) = P_t(\mathbf{x}_t, f_t, q_t) + H_t(\mathbf{x}_t, f_t, q_t) + W_t(\mathbf{x}_t, f_t, q_t)$. Consider the following three cases. First, suppose $q_t^{TB} = q_t^B$. Then clearly, $\Gamma_t(\mathbf{x}_t^{TB}, f_t, q_t^{TB}) = \Gamma_t(\mathbf{x}_t^{TB}, f_t, q_t^B)$. Second, suppose $q_t^{TB} > q_t^B$. Then we have $q_t^{TB} = q_t^L$. Given $\mathbf{x}_t$ and f_t, it is straightforward to check that $\Gamma_t(\mathbf{x}_t, f_t, q_t)$ is convex in q_t. Further, since $q_t^{TB} = q_t^L$ minimizes $\Gamma_t(\mathbf{x}_t^{TB}, f_t, q_t)$, we must have

$\Gamma_t(\mathbf{x}_t^{TB}, f_t, q_t^{TB}) \leq \Gamma_t(\mathbf{x}_t^{TB}, f_t, q_t^B)$ (this is why the lower bound q_t^L in the definition of policy TB cannot be replaced by any tighter ones). Last, suppose $q_t^{TB} < q_t^B$. Then we have $q_t^{TB} = q_t^U$. Since $\Gamma_t(\mathbf{x}_t^{TB}, f_t, q_t)$ is convex in q_t, q_t^L minimizes $\Gamma_t(\mathbf{x}_t^{TB}, f_t, q_t)$, and $q_t^B > q_t^{TB} \geq q_t^L$, we also have $\Gamma_t(\mathbf{x}_t^{TB}, f_t, q_t^{TB}) \leq \Gamma_t(\mathbf{x}_t^{TB}, f_t, q_t^B)$. By definition, for any given f_t, $\mathrm{E}[P_t^{TB} + H_t^{TB} + W_t^{TB}|f_t] = \Gamma_t(\mathbf{x}_t^{TB}, f_t, q_t^{TB})$, $\mathrm{E}[P_t^B + H_t^B + W_t^B|f_t] = \Gamma_t(\mathbf{x}_t^{TB}, f_t, q_t^B)$. Therefore:

$$\mathrm{E}[P_t^{TB} + H_t^{TB} + W_t^{TB}|f_t] \leq \mathrm{E}[P_t^B + H_t^B + W_t^B|f_t] \tag{A.4}$$

With Inequalities A.2-A.4, the remaining steps to prove $\mathrm{E}[\mathscr{C}(TB)] \leq 2\mathrm{E}[\mathscr{C}(IM)]$ follow the same way as before, which completes the proof. $\square$

Proof of Proposition 2. Due to Lemma 3, it remains to prove the "if" part of the proposition, i.e., if for $t = 1, ..., T$, $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1$, $\forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$, then Assumption 1 holds. First, since $C_{T+1}^{(k)}(\mathbf{x}_{T+1}, f_{T+1}) = 0, k = 1, ..., K-1, \forall \mathbf{x}_{T+1}, f_{T+1}$, issuing products of age $K-1$ at T clearly results in less cost than issuing younger products and letting the oldest products outdate. Also, how we issue products of age less than $K-1$ at T does not affect the total cost. Therefore, Assumption 1 holds for T.

Assume that Assumption 1 holds for $t+1$, i.e., at period $t+1$, if $y_{t+1} \leq \bar{y}_{t+1}$ and an optimal ordering decision rule is implemented at $t+2, ..., T$, then for any any demand $d_{t+1} \geq 0$ and future demand distribution defined by $f_{t+2} \in \mathscr{F}_{t+2}$, FIFO minimizes the future expected cost. We next show that the Assumption 1 also holds for t. At period t, given $\mathbf{x}_t$ and q_t such that $y_t \leq \bar{y}_t$, we must have $\sum_{k=1}^{K-1} x_{k,t+1} \leq \bar{y}_t$. Thus, under an optimal ordering decision rule, we have $y_{t+1} \leq \bar{y}_{t+1}$. Then, by induction assumption, FIFO is optimal for $t+1, ..., T$. It remains to show that FIFO is also optimal at period t. First, issuing products of age $K-1$ at period t results in less total cost than issuing younger products and letting the oldest products outdate because $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1$. Thus, an optimal issuing policy will issue as many oldest products as possible at period t. Let γ be such an issuing policy. Then, the costs that occur at period t by following FIFO and γ are exactly the same. Further, let $\mathbf{x}_{t+1}$ and $\mathbf{x}'_{t+1}$ be the inventory vectors at period $t+1$ by following FIFO an γ, respectively. Then, we have $\sum_{k=1}^{K-1} x_{k,t+1} = \sum_{k=1}^{K-1} x'_{k,t+1}$ and $\sum_{k=m}^{K-1} x_{k,t+1} \leq \sum_{k=m}^{K-1} x'_{k,t+1}, m = 2, ..., K-1$. From the proof of Lemma 3, we know that if for

$t = 1, ..., T$, $C^{(k)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1, \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t, \forall f_{t+1}$, then for $t = 1, ..., T$, $C^{(i)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq C^{(j)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}), 1 \leq i < j \leq K-1, \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t, \forall f_{t+1}$. Therefore, FIFO is also optimal at period t. □

Proof of Proposition 3. Due to Proposition 2, to show Assumption 1 holds, it is sufficient to show that for $t = 1, ..., T$, $C^{(k)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1, \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t, \forall f_{t+1}$.

The claim is clearly true for T since $C_{T+1}(\mathbf{x}_{T+1}, f_{T+1}) = 0, \forall \mathbf{x}_{T+1}, f_{T+1}$. Assume that the claim is true for $t, ..., T$. We now show that it is also true for $t-1$. At period t, given $\mathbf{x}_t$ and f_t such that $\sum_{k=1}^{K-1} x_{k,t} < \bar{y}_{t-1}$, consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}_t$ and System 2 starts from $\mathbf{x}'_t$, where $x'_{k,t} = x_{k,t} + \epsilon, x'_{m,t} = x_{m,t}, m \neq k, k = 1, ..., K-1$ (i.e., System 2 starts with ϵ more units of age k), and ϵ is positive but sufficiently small such that $\sum_{k=1}^{K-1} x_{k,t} + \epsilon \leq \bar{y}_t$. Let System 1 follow an optimal ordering decision rule, and let System 2 order up to the same level as System 1 at period t (order nothing if this is not feasible) and follow an optimal ordering decision rule afterward. Then, it is sufficient to show that the expected total cost in System 2 is at most $w\epsilon/\beta$ more than that in System 1.

Let y_t and y'_t be the total inventory levels after ordering at period t in Systems 1 and 2, respectively. Then, by construction, we have $y_t \leq y'_t \leq \bar{y}_t$. Let $\eta := y'_t - y_t =\leq \epsilon$. Since $\Phi_t(\bar{y}_t) \leq \gamma$, there will be at most $\gamma h \eta$ more expected holding cost and at least $(1-\gamma)p\eta$ less expected shortage penalty in System 2 than in System 1 at period t. For any demand realization d_t, let $\mathbf{x}_{t+1}$ and $\mathbf{x}'_{t+1}$ be the inventory vectors at period $t+1$ under Systems 1 and 2, respectively. Then, by construction, we have $x_{1,t+1} \geq x'_{1,t+1}, x_{k,t+1} \leq x'_{k,t+1}, k = 2, ..., K-1$. Assume that there are $\xi \leq \epsilon$ more units of outdates in System 2 than in System 1 at period t. Then $\sum_{k=2}^{K-1} x'_{k,t} - \sum_{k=2}^{K-1} x_{k,t} = \epsilon - \xi$. Since $0 \leq C^{(k)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 2, ..., K-1, \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t, \forall f_{t+1}$, the expected total cost in System 2 is at most $\gamma h \eta - (1-\gamma)p\eta + w\epsilon + \beta(\epsilon - \xi)w/\beta \leq w\epsilon/\beta$ more than that in System 1, where the inequality holds because of $h \leq \frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$. □

APPENDIX B

APPENDIX FOR CHAPTER 3

B.1 Cost Transformation

In this section, we consider a problem with a positive unit ordering costs c^i for each location $i = 1, 2$. We also consider an end-of-horizon salvage value at each location i which we assume is the same as the ordering cost c^i. In the following, we conduct a cost transformation to construct a new problem instance with zero ordering cost, and we show that the transformed problem is equivalent to the original problem with positive ordering costs. In particular, for $i = 1, 2$, define the cost parameters for the transformed problem as $\hat{c}^i = 0, \hat{p}^i = p^i - c^i, \hat{h}^i = h^i + c^i - \beta c^i, \hat{w}^i = w^i + \beta c^i$, and $\hat{r}^i = r^i + c^i - c^{-i}$. We next show that the original and transformed problems are equivalent, i.e., they have the same set of optimal policies.

Let $z_t^i = \sum_{k=1}^{K-1} x_{k,t}^i$ be the total inventory level at location i at period t before ordering. Since we assume that the system starts with zero inventory, we must have $z_t^i \leq S^i, i = 1, 2$. Then, the total costs of the original and transformed problems at period t are as follows:

$$L_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t) = c^1(S^1 - z_t^1) + c^2(S^2 - z_t^2) + p^1(d_t^1 + u_t - S^1)^+ + p^2(d_t^2 - u_t - S^2)^+ + h^1(S^1 - d_t^1 - u_t)^+$$

$$+ h^2(S^2 - d_t^2 + u_t)^+ + w^1(x_{K-1,t}^1 - d_t^1 - u_t)^+ + w^2(x_{K-1,t}^2 - d_t^2 + u_t)^+ + r^1(u_t)^+ + r^2(-u_t)^+,$$

$$\hat{L}_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t) = \hat{p}^1(d_t^1 + u_t - S^1)^+ + \hat{p}^2(d_t^2 - u_t - S^2)^+ + \hat{h}^1(S^1 - d_t^1 - u_t)^+ + \hat{h}^2(S^2 - d_t^2 + u_t)^+$$

$$+ \hat{w}^1(x_{K-1,t}^1 - d_t^1 - u_t)^+ + \hat{w}^2(x_{K-1,t}^2 - d_t^2 + u_t)^+ + \hat{r}^1(u_t)^+ + \hat{r}^2(-u_t)^+.$$

For a given sample path, the total costs of the original and transformed problems are $\sum_{t=1}^{T} \beta^{t-1} L_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t) - \beta^T(c^1 z_{t+1}^1 + c^2 z_{t+1}^2)$ and $\sum_{t=1}^{T} \beta^{t-1} \hat{L}_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)$ respectively. To show that the original and transformed problems have the same optimal policy, it is sufficient to show that the difference between the costs of the two problems is a constant independent of the ordering and transshipment policies.

For location 1, we have $(d_t^1 + u_t - S^1)^+ - (S^1 - d_t^1 - u_t)^+ = d_t^1 + u_t - S^1$, and $(S^1 - d_t^1 -$

$u_t)^+ - (x^1_{K-1,t} - d^1_t - u_t)^+ = z^1_{t+1}$. Similarly, for location 2, we have $(d^2_t - u_t - S^2)^+ - (S^2 - d^2_t + u_t)^+ = d^2_t - u_t - S^2$, and $(S^2 - d^2_t + u_t)^+ - (x^2_{K-1,t} - d^2_t + u_t)^+ = z^2_{t+1}$. Finally, we have $(u_t)^+ - (-u_t)^+ = u_t$. Therefore, the cost difference between the two problems is:

$$
\begin{aligned}
& \sum_{t=1}^{T} \beta^{t-1} \hat{L}_t(\mathbf{x}^1_t, \mathbf{x}^2_t, u_t) - \sum_{t=1}^{T} \beta^{t-1} L_t(\mathbf{x}^1_t, \mathbf{x}^2_t, u_t) + \beta^T (c^1 z^1_{t+1} + c^2 z^2_{t+1}) \\
= \;& \sum_{t=1}^{T} \beta^{t-1} \Big(- c^1(S^1 - z^1_t) - c^2(S^2 - z^2_t) - c^1(d^1_t + u_t - S^1)^+ - c^2(d^2_t - u_t - S^2)^+ \\
& + (c^1 - \beta c^1)(S^1 - d^1_t - u_t)^+ + (c^2 - \beta c^2)(S^2 - d^2_t + u_t)^+ + \beta c^1 (x^1_{K-1,t} - d^1_t - u_t)^+ \\
& + \beta c^2 (x^2_{K-1,t} - d^2_t + u_t)^+ + (c^1 - c^2)(u_t)^+ + (c^2 - c^1)(-u_t)^+ \Big) + \beta^T (c^1 z^1_{t+1} + c^2 z^2_{t+1}) \\
= \;& \sum_{t=1}^{T} \beta^{t-1} \Big(- c^1(S^1 - z^1_t) - c^2(S^2 - z^2_t) - c^1(d^1_t + u_t - S^1) - c^2(d^2_t - u_t - S^2) \\
& - \beta c^1 z^1_{t+1} - \beta c^2 z^2_{t+1} + (c^1 - c^2) u_t \Big) + \beta^T (c^1 z^1_{t+1} + c^2 z^2_{t+1}) \\
= \;& c^1 z^1_1 + c^2 z^2_1 - \sum_{t=1}^{T} \beta^{t-1} d^1_t - \sum_{t=1}^{T} \beta^{t-1} d^2_t,
\end{aligned}
$$

where $z^i_1, i = 1, 2$ are the initial inventory levels and d^i_t are the demands at the two locations, all of which are independent of the ordering and transshipment policies. That is, the difference between the costs of the original and transformed problems is a constant. Therefore, these two problems must have the same set of optimal policies, which completes the proof.

B.2 Proofs of Analytical Results

Proof of Lemma 7. Due to symmetry, it is sufficient to prove the lemma for location 1. The conclusion clearly holds for $t = T + 1$ since $C_{T+1}(\mathbf{x}^1_{T+1}, \mathbf{x}^2_{T+1}) = 0, \forall \mathbf{x}^1_{T+1}, \mathbf{x}^2_{T+1}$. Assume that the conclusion holds for $t + 1$. We now show that it also holds for t. We prove this result in the following three steps.

First, we show that $C^{(K-1)}_t(\mathbf{x}^1_t, \mathbf{x}^2_t) \le w^1, k = 1, ..., K - 1$. For any given $(\mathbf{x}^1_t, \mathbf{x}^2_t)$ such that $\sum_{k=1}^{K-1} x^1_{k,t} < S^1, \sum_{k=1}^{K-1} x^2_{k,t} < S^2$, let $\epsilon > 0$ be a sufficiently small number such that $\sum_{k=1}^{K-1} x^1_{k,t} + \epsilon \le S^1$. Consider the following two systems in parallel: System A starting with initial inventory $(\mathbf{x}^{1A}_t, \mathbf{x}^{2A}_t)$ and System B starting with $(\mathbf{x}^{1B}_t, \mathbf{x}^{2B}_t)$, where $\mathbf{x}^{1A}_t = \mathbf{x}^1_t, x^{1B}_{K-1,t} = x^1_{K-1,t} + \epsilon, x^{1B}_{j,t} =$

$x^1_{j,t}, j \neq K-1$, and $\mathbf{x}^{2A}_t = \mathbf{x}^{2B}_t = \mathbf{x}^2_t$, i.e., System B has ϵ more units of age $K-1$ product at location 1 than System A, while the two systems have the same inventory levels of all ages at location 2. Let both systems follow the same base-stock levels S^1 and S^2. Further, let System A follow an optimal transshipment policy, and let System B follow the same transshipment quantity as System A at each period (this is possible because the two systems have the same base-stock levels). Then, to prove $C^{(K-1)}_t(\mathbf{x}^1_t, \mathbf{x}^2_t) \leq w^1$, it is sufficient to show that the expected total cost of System B is at most $w^1\epsilon$ more than that of System A. By construction, the inventory at location 2 in the two systems will be the same in all periods, but it may not be the case for location 1. Assume that there are $\xi \in [0, \epsilon]$ more units of outdates at location 1 at t in System B than in System A. Then, by construction, we have $x^{1B}_{k,t+1} \geq x^{1A}_{k,t+1}, k = 2, ..., K-1$, and $\sum_{k=2}^{K-1} x^{1B}_{k,t+1} - \sum_{k=2}^{K-1} x^{1A}_{k,t+1} \leq \epsilon - \xi$. By induction assumption, the expected total cost of System B is at most $w^1\xi + \beta(\epsilon - \xi) \times w^1 \leq w^1\epsilon$ more than that of System A.

Second, we show that $C^{(j)}_t(\mathbf{x}^1_t, \mathbf{x}^2_t) \leq C^{(k)}_t(\mathbf{x}^1_t, \mathbf{x}^2_t), 1 \leq j < k \leq K-1$. For any given $(\mathbf{x}^1_t, \mathbf{x}^2_t)$ such that $\sum_{k=1}^{K-1} x^1_{k,t} < S^1, \sum_{k=1}^{K-1} x^2_{k,t} < S^2$, and any $1 \leq j < k \leq K-1$, let $\epsilon > 0$ be a sufficiently small number such that $\sum_{k=1}^{K-1} x^1_{k,t} + \epsilon \leq S^1$. Consider the following two systems: System A starting with $(\mathbf{x}^{1A}_t, \mathbf{x}^{2A}_t)$ and System B starting with $(\mathbf{x}^{1B}_t, \mathbf{x}^{2B}_t)$, where $x^{1A}_{j,t} = x^1_{j,t} + \epsilon, x^{1A}_{l,t} = x^1_{l,t}, l \neq j, x^{1B}_{k,t} = x^1_{k,t} + \epsilon, x^{1B}_{l,t} = x^1_{l,t}, l \neq k$, and $\mathbf{x}^{2A}_t = \mathbf{x}^{2B}_t = \mathbf{x}^2_t$, i.e., System A has ϵ more units of age j product while System B has ϵ more unit of age k product at location 1, and the two systems have the same inventory levels of all ages at location 2. Let both systems follow the same base-stock levels S^1 and S^2. Further, let System B follow an optimal transshipment policy, and let System A follow the same transshipment quantity as System B at each period. Then, it is sufficient to show that the expected total cost of System A is no more than that of System B. Clearly, the inventory at location 2 in the two systems will be the same at all periods. For location 1, assume that there are $\xi \in [0, \epsilon]$ more units of outdates at location 1 in System B than in System A (note that ξ can only be positive when $k = K-1$). Then, by construction, we have $\sum_{k=1}^{K-1} x^{1A}_{k,t+1} - \sum_{k=1}^{K-1} x^{1B}_{k,t+1} = \xi$, and $\sum_{k=1}^{l} x^{1A}_{k,t+1} \geq \sum_{k=1}^{l} x^{1B}_{k,t+1}, l = 1, ..., K-1$. Since $-w^1\xi + \beta\xi \times w^1 \leq 0$, by induction assumption, the expected total cost of System A is no more than that of System B.

Finally, we show that $C^{(1)}_t(\mathbf{x}^1_t, \mathbf{x}^2_t) \geq 0$. For any given $(\mathbf{x}^1_t, \mathbf{x}^2_t)$ such that $\sum_{k=1}^{K-1} x^1_{k,t} <$

$S^1, \sum_{k=1}^{K-1} x_{k,t}^2 < S^2$, let $\epsilon > 0$ be a sufficiently small number such that $\sum_{k=1}^{K-1} x_{k,t}^1 + \epsilon \leq S^1$. Consider the following two systems: System A starting with initial inventory $(\mathbf{x}_t^{1A}, \mathbf{x}_t^{2A})$ and System B starting with $(\mathbf{x}_t^{1B}, \mathbf{x}_t^{2B})$, where $\mathbf{x}_t^{1A} = \mathbf{x}_t^1, x_{1,t}^{1B} = x_{1,t}^1 + \epsilon, x_{j,t}^{1B} = x_{j,t}^1, j = 2, ..., K-1$, and $\mathbf{x}_t^{2A} = \mathbf{x}_t^{2B} = \mathbf{x}_t^2$, i.e., System B has ϵ more units of age 1 product at location 1 than System A, while the two systems have the same inventory levels of all ages at location 2. Let both systems follow the same base-stock levels S^1 and S^2. Further, let System B follow an optimal transshipment policy, and let System A follow the same transshipment quantity as System B at each period. Then, it is sufficient to show that the total cost of System A is no more than that of System B. Clearly, the inventory at location 2 in the two systems will be the same at all periods. Further, the total cost of period t in the two systems is the same and we have $\sum_{k=1}^{K-1} x_{k,t+1}^{1A} = \sum_{k=1}^{K-1} x_{k,t+1}^{1B}$ and $\sum_{k=1}^{l} x_{k,t+1}^{1A} \geq \sum_{k=1}^{l} x_{k,t+1}^{1B}, l = 1, ..., K-2$. By induction assumption, the expected total cost of System A is no more than that of System B, which completes the proof. □

Proof of Proposition 4. Define:

$$G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t) := L_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t) + \beta C_{t+1}(\mathbf{x}_{t+1}^1, \mathbf{x}_{t+1}^2).$$

Then, we have $C_t(\mathbf{x}_t^1, \mathbf{x}_t^2) = \mathrm{E}\left[\min_{-d_t^1 \leq u_t \leq d_t^2} \left\{ G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t) \right\}\right]$. Due to symmetry, it is sufficient to consider regions 1-5 and region 7.

(i) $d_t^1 < S^1, d_t^2 > S^2$ (i.e., regions 1 and 2). Suppose $u_t \leq 0$, i.e., demands (d_t^1, d_t^2) move in the northwest direction or do not move. Consider the following two cases:

Case 1: $d_t^1 + u_t < x_{K-1,t}^1$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into region 1). Then, increasing one unit of u_t will decrease one unit of outdate at location 1 and one unit of shortage at location 2, but possibly increase one unit of transshipment from location 1 to 2 (will increase if $u_t = 0$). Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t \leq -h^1 - w^1 - p^2 + r^1 < 0,$$

where $\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t$ denotes the right partial derivative of $G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)$ with respect to u_t, and should be interpreted as $G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t + 1) - G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)$ for the discrete case.

Case 2: $d_t^1 + u_t \geq x_{K-1,t}^1$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into region 2). Then, increasing one unit of u_t will decrease one unit of surplus inventory of age $k < K-1$ at location 1 (which becomes of age $k+1$ at period $t+1$) and one unit of shortage at location 2, but possibly increase one unit of transshipment from location 1 to 2 (will increase if $u_t = 0$). Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t \leq -h^1 - \beta C_{t+1}^{(k+1)}(\mathbf{x}_{t+1}^1, \mathbf{x}_{t+1}^2) - p^2 + r^1 \leq -h^1 - p^2 + r^1 < 0.$$

Combining the above two cases, we have $u_t^* > 0$.

(ii) $d_t^1 \geq S^1, d_t^2 \geq S^2$ (i.e., region 3). Suppose $u_t < 0$, i.e., demands (d_t^1, d_t^2) move in the northwest direction. Consider the following two cases:

Case 1: $d_t^1 + u_t < S^1$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into regions 1,2). Then, increasing one unit of u_t will decrease one unit of surplus inventory (and possibly also outdate) at location 1, one unit of shortage at location 2, and one unit of transshipment from location 2 to 1. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t \leq -h^1 - p^2 - r^2 < 0.$$

Case 2: $d_t^1 + u_t \geq S^1$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into region 3). Then, increasing one unit of u_t will increase one unit of shortage at location 1, but decrease one unit of shortage at location 2 and one unit of transshipment from location 2 to 1. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t = p^1 - p^2 - r^2 \leq 0.$$

Combining the above two cases, we have $u_t^* \geq 0$ (recall that in the case of multiple optimal solutions, u_t^* is defined as the one with the smallest magnitude). Due to symmetry, we also have $u_t^* \leq 0$. Therefore, $u_t^* = 0$.

(iii) $d_t^1 \leq x_{K-1,t}^1, x_{K-1,t}^2 < d_t^2 \leq S^2$ (i.e., region 4). Suppose $u_t < 0$, i.e., demands (d_t^1, d_t^2) move in the northwest direction. Consider the following two cases:

Case 1: $d_t^2 - u_t > S^2$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into region 1). Then, increasing one unit of u_t will decrease one unit of outdate at location 1, one unit of shortage at location 2, and one unit of transshipment from location 2 to 1. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t = -h^1 - w^1 - p^2 - r^2 < 0.$$

Case 2: $d_t^2 - u_t \leq S^2$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into region 4). Then, increasing one unit of u_t will decrease one unit of outdate at location 1, increase one unit of surplus inventory of age $k < K-1$ at location 2 (which becomes of age $k+1$ at period $t+1$), and decrease one unit of transshipment from location 2 to 1. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t = -h^1 - w^1 + h^2 + \beta \hat{C}_{t+1}^{(k+1)}(\mathbf{x}_{t+1}^1, \mathbf{x}_{t+1}^2) - r^2 \leq -h^1 - w^1 + h^2 + w^2 - r^2 \leq 0.$$

Combining the above two cases, we have $u_t^* \geq 0$.

(iv) $x_{K-1,t}^1 < d_t^1 \leq S^1, x_{K-1,t}^2 < d_t^2 \leq S^2$ (i.e., region 5). To prove the conclusion for this region, it is sufficient to construct problem instances where the optimal transshipment is positive, negative and zero, respectively. Consider an instance where the transshipment costs $r^i = 0, i = 1, 2$. Then, the system becomes equivalent to a single-location system, where FIFO is the optimal issuing policy when inventory is replenished based on a base-stock policy [15]. In this case, inventory should be transshipped from location 1 (2) to location 2 (1) when the age of the oldest product after meeting demand but before transshipment at location 1 is larger (smaller) than that at location 2, and no transshipment is needed when the age of the oldest product after meeting demand but before transshipment at the two locations is the same.

(v) $d_t^1 \leq x_{K-1,t}^1, d_t^2 \leq x_{K-1,t}^2$ (i.e., region 7). Suppose $u_t < 0$, i.e., demands (d_t^1, d_t^2) move in the northwest direction. Consider the following three cases:

Case 1: $d_t^2 - u_t > S^2$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into region 1). Then, increasing one unit of u_t will decrease one unit of outdate at location 1, one unit of shortage at location 2, and one unit of transshipment cost for transshipment from location 2 to 1. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t = -h^1 - w^1 - p^2 - r^2 < 0.$$

Case 2: $x_{K-1,t}^2 < d_t^2 - u_t \leq S^2$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into region 4). Then, increasing one unit of u_t will decrease one unit of outdate at location 1, increase one unit surplus inventory of age $k < K-1$ at location 2 (which becomes of age $k+1$ at period $t+1$), and decrease one unit of transshipment cost for transshipment from location 2 to 1. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t = -h^1 - w^1 + h^2 + \beta \hat{C}_{t+1}^{(k+1)}(\mathbf{x}_{t+1}^1, \mathbf{x}_{t+1}^2) - r^2 \leq -h^1 - w^1 + h^2 + w^2 - r^2 \leq 0.$$

Case 3: $d_t^2 - u_t \leq x_{K-1,t}^2$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into region 7). Then, increasing one unit of u_t will decrease one unit of outdate at location 1, increase one unit of outdate at location 2, and decrease one unit of transshipment cost for transshipment from location 2 to 1. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t = -h^1 - w^1 + h^2 + w^2 - r^2 \leq 0.$$

Combining the above three cases, we have $u_t^* \geq 0$. Due to symmetry, we also have $u_t^* \leq 0$. Therefore, $u_t^* = 0$.

Combining all scenarios (i)-(v), the conclusion in Proposition 4 holds. □

Proof of Proposition 5. Due to symmetry, it is sufficient to prove that $u_t^* \leq \min\{(S^1 - d_t^1)^+, (d_t^2 - x_{K-1,t}^2)^+\}$. Then, it is sufficient to consider the case where $d_t^1 \leq S^1, d_t^2 \geq x_{K-1,t}^2$ (i.e., regions 1,2,4,5), since in other regions, we have already shown in Proposition 4 that $u_t^* \leq 0$. In regions 1,2,4,5, suppose $u_t > \min\{(S^1 - d_t^1)^+, (d_t^2 - x_{K-1,t}^2)^+\}$. Consider the following three cases:

Case 1: $d_t^1 + u_t > S^1, d_t^2 - u_t > S^2$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into region 3). Then, increasing one unit of u_t will increase one unit of shortage at location 1, decrease one unit of shortage at location 2, and increase one unit of transshipment from location 1 to 2. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t = p^1 - p^2 + r^1 \geq 0.$$

Case 2: $d_t^1 + u_t > S^1, d_t^2 - u_t \leq S^2$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into regions 6,9). Then, increasing one unit of u_t will increase one unit of shortage at location 1, one unit of surplus inventory (and possibly also outdate), and one unit of transshipment from location 1 to 2. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t \geq p^1 + h^2 + r^1 > 0.$$

Case 3: $d_t^1 + u_t \le S^1, d_t^2 - u_t < x_{K-1,t}^2$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into regions 7,8). Then, increasing one unit of u_t will decrease one unit of surplus inventory (and possibly also outdate) at location 1, but increase one unit of outdate at location 2 and one unit of transshipment from location 1 to 2. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t \ge -h^1 - w^1 + h^2 + w^2 + r^1 \ge 0.$$

Combining the above three cases, we have $u_t^* \le \min\{(S^1 - d_t^1)^+, (d_t^2 - x_{K-1,t}^2)^+\}$ (recall that in the case of multiple optimal solutions, u_t^* is defined as the one with the smallest magnitude). □

Proof of Proposition 6. Due to symmetry, it is sufficient to consider the case where $d_t^1 \le S^1, d_t^2 \ge x_{K-1,t}^2$ (i.e., regions 1,2,4,5). Consider the following two scenarios:

(i) $d_t^1 < S^1, d_t^2 > S^2$ (i.e., regions 1,2). Suppose $u_t \neq \min\{S^1 - d_t^1, d_t^2 - S^2\}$. Consider the following four cases:

Case 1: $u_t < \min\{S^1 - d_t^1, d_t^2 - S^2\}$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into regions 1,2). Then, increasing one unit of u_t will decrease one unit of surplus inventory (and possibly also outdate) at location 1 and one unit of shortage at location 2, but possibly increase one unit of transshipment from location 1 to 2 (will increase if $u_t \ge 0$). Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t \le -h^1 - p^2 + r^1 < 0.$$

Case 2: $S^1 - d_t^1 < u_t < d_t^2 - S^2$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into region 3). Then, increasing one unit of u_t will increase one unit of shortage at location 1, decrease one unit of shortage at location 2, and increase one unit of transshipment from location 1 to 2. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t = p^1 - p^2 + r^1 \ge 0.$$

Case 3: $d_t^2 - S^2 < u_t < S^1 - d_t^1$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into regions 4,5,7,8). Then, increasing one unit of u_t will decrease one unit of surplus inventory (and possibly also outdate) at location 1, but increase one unit of surplus inventory (and possibly also outdate) at location 2 and one unit of transshipment from location 1 to 2. The cost of increasing one unit of surplus inventory at location i is bounded by $h^i + w^i$ (no matter whether this unit is outdated at the current period or is carried to the next period). Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t \geq -h^1 - w^1 + h^2 + r^1 \geq 0.$$

Case 4: $u_t \geq \max\{S^1 - d_t^1, d_t^2 - S^2\}$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into regions 6,9). Then, increasing one unit of u_t will increase one unit of shortage at location 1, one unit of surplus inventory (and possibly also outdate) at location 2, and one unit of transshipment from location 1 to 2. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t \geq p^1 + h^2 + r^1 > 0.$$

Combining the above four cases, we have $u_t^* = \min\{S^1 - d_t^1, d_t^2 - S^2\}$.

(ii) $d_t^1 \leq S^1, x_{K-1,t}^2 \leq d_t^2 \leq S^2$ (i.e., regions 4,5). Suppose $u_t \neq 0$. Consider the following four cases:

Case 1: $u_t < d_t^2 - S^2$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into regions 1,2). Then, increasing one unit of u_t will decrease one unit of surplus inventory (and possibly also outdate) at location 1, one unit of shortage at location 2, and one unit of transshipment from location 2 to 1. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t \leq -h^1 - p^2 - r^2 < 0.$$

Case 2: $d_t^2 - S^2 \leq u_t < 0$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into regions 4,5 and $u_t < 0$). Then, increasing one unit of u_t will decrease one unit of surplus inventory (and possibly also outdate) at location 1, increase one unit of surplus inventory at location 2, and decrease one unit of transshipment from location 2 to 1. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t \leq -h^1 + h^2 - r^2 \leq 0.$$

Case 3: $0 \leq u_t \leq S^1 - d_t^1$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into regions 4,5,7,8 and $u_t > 0$). Then, increasing one unit of u_t will decrease one unit of surplus inventory (and possibly also outdate) at location 1, but increase one unit of surplus inventory (and possibly also outdate) at location 2 and one unit of transshipment from location 1 to 2. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t \geq -h^1 - w^1 + h^2 + r^1 \geq 0.$$

Case 4: $u_t > S^1 - d_t^1$ (i.e., $(d_t^1 + u_t, d_t^2 - u_t)$ fall into regions 6,9). Then, increasing one unit of u_t will increase one unit of shortage at location 1, one unit of surplus inventory (and possibly also outdate) at location 2, and one unit of transshipment from location 1 to 2. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t \geq p^1 + h^2 + r^1 > 0.$$

Combining the above four cases, we have $u_t^* = 0$. $\square$

Proof of Proposition 7. Due to symmetry, it is sufficient to consider the case where $d_t^1 \leq S^1, d_t^2 \geq x_{K-1,t}^2$ (i.e., regions 1,2,4,5). There is nothing to prove in regions 4 and 5 because we always have $|u_t^*| \geq 0$. Therefore, it is sufficient to prove that $u_t^* \geq \min\{S^1 - d_t^1, d_t^2 - S^2\}$ when $d_t^1 < S^1, d_t^2 > S^2$ (i.e., regions 1,2), the proof for which follows from Scenario (i) Case 1 of the proof for Proposition 6. $\square$

Proof of Theorem 3. The claim is clearly true for $T+1$ since $\tilde{C}_{T+1}(s_{T+1}^1, s_{T+1}^2) = 0, \forall s_{T+1}^1, s_{T+1}^2$. Assume that the claim is true for $t+1$, we now show that it is also true for t. Define:

$$\tilde{G}_t(s_t^1, s_t^2, u_t) := L_t(x_t^1, x_t^2, u_t) + \beta C_{t+1}(x_{t+1}^1, x_{t+1}^2).$$

Then:

$$\tilde{C}_t(s_t^1, s_t^2) = \mathrm{E}\left[\min_{-d_t^1 \leq u_t \leq d_t^2} \left\{\tilde{G}_t(s_t^1, s_t^2, u_t)\right\}\right].$$

Since $L^\natural$-convexity is preserved i) under minimization when each of the constraints involves either only one variable or two variables with opposite signs, and ii) under expectation [77, 82], to show that $\tilde{C}_t(s_t^1, s_t^2)$ is $L^\natural$-convex in (s_t^1, s_t^2), it suffices to show that $\tilde{G}_t(s_t^1, s_t^2, u_t)$ is $L^\natural$-convex in (s_t^1, s_t^2, u_t).

In order to do so, we next reformulate the problem of each period into a two-stage problem. In particular, we assume that in addition to the ordering and transshipment quantities, we also need to determine how many units issue from inventory (either used for meeting demand or simply disposed from inventory with an outdating cost w^i). Suppose given demand realization $d_t^i, i = 1, 2$, we issue (either to meet demand or dispose) ξ_t^1 units out of $S^1 - u_t$, and ξ_t^2 units out of $S^2 + u_t$. To ensure that we dispose all the outdated units but dispose no more than we have in hand, we need the following constraints:

$$\begin{cases} s_t^1 - u_t \leq \xi_t^1 \leq S^1 - u_t, \\ -s_t^2 + u_t \leq \xi_t^2 \leq S^2 + u_t. \end{cases}$$

From Lemma 7, we know that one additional unit in inventory increases the expected total cost, but the cost is bounded by w^i. Therefore, the reformulation will give us exactly the same solution as the original problem, i.e., we will issue the products to satisfy as much as demand as possible, and will issue no more than what is needed to meet demand plus what is outdated. Let $\eta_t^1 = -S^1 + u_t + \xi_t^1$, and $\eta_t^2 = S^2 + u_t - \xi_t^2$. Then, the above constraints become as follows:

$$\begin{cases} s_t^1 - S^1 \leq \eta_t^1 \leq 0, \\ 0 \leq \eta_t^2 \leq s_t^2 + S^2. \end{cases}$$

Further, the system dynamics can be described as follows:

$$s_{t+1}^1 = S^1 - u_t - \xi_t^1 = -\eta_t^1,$$

$$s_{t+1}^2 = -(S^2 + u_t - \xi_t^2) = -\eta_t^2.$$

Define:

$$\begin{aligned}
&\tilde{L}_t(s_t^1, s_t^2, u_t, \eta_t^1, \eta_t^2) \\
:=&p^1(d_t^1 - \xi_t^1)^+ + p^2(d_t^2 - \xi_t^2)^+ + h^1(S^1 - u_t - \xi_t^1) + h^2(S^2 + u_t - \xi_t^2) \\
&+ (h^1 + w^1)(\xi_t^1 - d_t^1)^+ + (h^2 + w^2)(\xi_t^2 - d_t^2)^+ + r^1(u_t)^+ + r^2(-u_t)^+ \\
=&p^1(d_t^1 - S^1 + u_t - \eta_t^1)^+ + p^2(d_t^2 - S^2 - u_t + \eta_t^2)^+ + h^1(-\eta_t^1) + h^2\eta_t^2 \\
&+ (h^1 + w^1)(S^1 - u_t + \eta_t^1 - d_t^1)^+ + (h^2 + w^2)(S^2 + u_t - \eta_t^2 - d_t^2)^+ + r^1(u_t)^+ + r^2(-u_t)^+.
\end{aligned}$$

Then, we have:

$$\tilde{G}_t(s_t^1, s_t^2, u_t) = \min_{\eta_t^1, \eta_t^2} \left\{ \tilde{L}_t(s_t^1, s_t^2, u_t, \eta_t^1, \eta_t^2) + \beta \tilde{C}_{t+1}(s_{t+1}^1, s_{t+1}^2) \right\}.$$

Clearly, $\tilde{L}_t(s_t^1, s_t^2, u_t, \eta_t^1, \eta_t^2)$ is $L^\natural$-convex in $(s_t^1, s_t^2, u_t, \eta_t^1, \eta_t^2)$ because each term involves either only one variable or two variables with opposite signs. Further, by Lemma 6 in [82], $\tilde{C}_{t+1}(s_{t+1}^1, s_{t+1}^2) = \tilde{C}_{t+1}(-\eta_t^1, -\eta_t^2)$ is also $L^\natural$-convex in $(s_t^1, s_t^2, u_t, \eta_t^1, \eta_t^2)$. Since $L^\natural$-convexity is preserved under minimization when each of the constraints involves either only one variable or two variables with opposite signs, $\tilde{G}_t(s_t^1, s_t^2, u_t)$ is $L^\natural$-convex in (s_t^1, s_t^2, u_t), which concludes the proof. □

Proof of Proposition 8. Due to symmetry, it is sufficient to consider the case where $d_t^1 \leq S^1, d_t^2 \geq x_{K-1,t}^2$ (i.e., regions 1,2,4,5). There is nothing to prove in region 5. Further, given Proposition 7, we know that when (d_t^1, d_t^2) lie in regions 1,2, the demand after transshipment $(d_t^1 + u_t, d_t^2 - u_t)$ will at least hit the boundary between regions 1,2 and regions 3,4 and 5. Then, it remains to prove that $u_t^* \geq \hat{u}_t$ in region 4.

When $d_t^1 \leq x_{K-1,t}^1, x_{K-1,t}^2 < d_t^2 \leq S^2$ (i.e., region 4), we know that $u_t^* \geq 0$ (Proposition 4). Suppose $0 \leq u_t < \hat{u}_t = \min\{x_{K-1,t}^1 - d_t^1, d_t^2 - x_{K-1,t}^2\}$. Then, $(d_t^1 + u_t, d_t^2 - u_t)$ still lie in region 4. In this case, increasing one unit of u_t will decrease one unit of outdate at location 1, but increase one unit of surplus inventory of age $k < K - 1$ at location 2 (which becomes of age $k + 1$ at period $t + 1$) and one unit of transshipment from location 1 to 2. Therefore:

$$\partial G_t(\mathbf{x}_t^1, \mathbf{x}_t^2, u_t)/\partial u_t = -h^1 - w^1 + h^2 + \beta \hat{C}_{t+1}^{(k+1)}(\mathbf{x}_{t+1}^1, \mathbf{x}_{t+1}^2) + r^1 \leq -h^1 - w^1 + h^2 + \beta w^2 + r^1 < 0.$$

Then, we have $u_t^* \geq \hat{u}_t = \min\{x_{K-1,t}^1 - d_t^1, d_t^2 - x_{K-1,t}^2\}$, which completes the proof. □

B.3 Numerical Results for Larger Lifetimes

Table B.1: Expected total costs under different policies: symmetric case (percentages of cost reduction over policy NS are included in parenthesis).

Policy			NS	S1	S2	Our Policy
Poisson	$K = 4$	$p = 5, h = 0.5$	82.1	67.1 (18.2%)	63.1 (23.1%)	63.1 (23.1%)
		$p = 5, h = 1$	136.2	104.8 (23.1%)	102.6 (24.6%)	102.6 (24.6%)
		$p = 10, h = 0.5$	93.7	82.4 (12.1%)	70.5 (24.8%)	70.4 (24.9%)
		$p = 10, h = 1$	164.0	130.8 (20.2%)	119.9 (26.9%)	119.9 (26.9%)
	$K = 6$	$p = 5, h = 0.5$	81.9	67.0 (18.2%)	63.1 (23.0%)	63.1 (23.0%)
		$p = 5, h = 1$	136.1	104.8 (23.1%)	102.6 (24.6%)	102.6 (24.6%)
		$p = 10, h = 0.5$	93.3	82.0 (12.1%)	70.4 (24.6%)	70.4 (24.6%)
		$p = 10, h = 1$	163.8	130.7 (20.2%)	119.9 (26.8%)	119.9 (26.8%)
Geometric	$K = 4$	$p = 5, h = 0.5$	299.3	203.0 (32.2%)	200.0 (33.2%)	191.5 (36.0%)
		$p = 5, h = 1$	396.9	280.1 (29.4%)	280.1 (29.4%)	274.4 (30.9%)
		$p = 10, h = 0.5$	400.7	278.5 (30.5%)	243.2 (39.3%)	229.7 (42.7%)
		$p = 10, h = 1$	544.5	368.6 (32.3%)	352.6 (35.2%)	342.6 (37.1%)
	$K = 6$	$p = 5, h = 0.5$	248.5	182.9 (26.4%)	171.5 (31.0%)	169.5 (31.8%)
		$p = 5, h = 1$	368.1	267.2 (27.4%)	265.4 (27.9%)	264.7 (28.1%)
		$p = 10, h = 0.5$	311.0	240.0 (22.8%)	199.1 (36.0%)	195.8 (37.0%)
		$p = 10, h = 1$	484.1	343.2 (29.1%)	321.2 (33.6%)	319.5 (34.0%)

Note. Other cost parameters are $w = 5, r = 1$.

Table B.2: Expected total costs under different policies: asymmetric case (percentages of cost reduction over policy NS are included in parenthesis).

Policy			NS	S1	S2	Our Policy
Poisson	$K = 4$	$p = 5, h = 0.5$	41.0	41.0 (0.0%)	41.0 (0.0%)	40.9 (0.2%)
		$p = 5, h = 1$	68.0	68.0 (0.0%)	68.0 (0.0%)	68.0 (0.0%)
		$p = 10, h = 0.5$	46.8	46.8 (0.0%)	46.8 (0.0%)	46.7 (0.4%)
		$p = 10, h = 1$	81.9	81.9 (0.0%)	81.9 (0.0%)	81.9 (0.1%)
	$K = 6$	$p = 5, h = 0.5$	40.9	40.9 (0.0%)	40.9 (0.0%)	40.9 (0.0%)
		$p = 5, h = 1$	68.0	68.0 (0.0%)	68.0 (0.0%)	68.0 (0.0%)
		$p = 10, h = 0.5$	46.6	46.6 (0.0%)	46.6 (0.0%)	46.6 (0.0%)
		$p = 10, h = 1$	81.9	81.9 (0.0%)	81.9 (0.0%)	81.9 (0.0%)
Geometric	$K = 4$	$p = 5, h = 0.5$	150.0	150.0 (0.0%)	143.8 (4.1%)	129.2 (13.9%)
		$p = 5, h = 1$	199.0	199.0 (0.0%)	199.3 (-0.2%)	186.7 (6.2%)
		$p = 10, h = 0.5$	200.6	200.6 (0.0%)	171.6 (14.5%)	157.0 (21.8%)
		$p = 10, h = 1$	272.7	272.7 (0.0%)	261.8 (4.0%)	247.2 (9.4%)
	$K = 6$	$p = 5, h = 0.5$	124.5	124.5 (0.0%)	124.9 (-0.4%)	119.6 (3.9%)
		$p = 5, h = 1$	184.5	184.5 (0.0%)	184.5 (0.0%)	182.8 (0.9%)
		$p = 10, h = 0.5$	155.6	155.6 (0.0%)	152.7 (1.9%)	146.2 (6.1%)
		$p = 10, h = 1$	242.5	242.5 (0.0%)	243.0 (-0.2%)	237.6 (2.0%)

Note. Other cost parameters are $w = 5, r = 1$.

APPENDIX C

APPENDIX FOR CHAPTER 4

C.1 Proofs of Analytical Results

Proof of Proposition 1. First, note that the existence of a truthful mechanism is trivial: If $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$ is equal to the same constant for all possible pairs of $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \Gamma$, then recipients do not have an incentive to misreport because i) their probability of being selected is the same regardless of what they report, and ii) when selected, truth-reporting maximizes their container value.

Next, we show that when the MSRO inventory information is shared with recipients (our analysis is implicitly built on this assumption since recipients compute expectations under known inventory levels), if a mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$ is truthful, then there must exist $\tilde{p}_1, \tilde{p}_2 \in [0, 1]$ such that $E_{\boldsymbol{\gamma}_2}[p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)] = \tilde{p}_1, \forall \boldsymbol{\gamma}_1 \in \Gamma$, and $E_{\boldsymbol{\gamma}_1}[1 - p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)] = \tilde{p}_2, \forall \boldsymbol{\gamma}_2 \in \Gamma$. Due to symmetry, it is sufficient to prove this result for recipient 1. Then, it suffices to show that $E_{\boldsymbol{\gamma}_2}[p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)] = E_{\boldsymbol{\gamma}_2}[p(\boldsymbol{\gamma}'_1, \boldsymbol{\gamma}_2)], \forall \boldsymbol{\gamma}_1, \boldsymbol{\gamma}'_1 \in \Gamma$. Suppose this is not true, i.e., $\exists \boldsymbol{\gamma}_1, \boldsymbol{\gamma}'_1$ such that $E_{\boldsymbol{\gamma}_2}[p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)] < E_{\boldsymbol{\gamma}_2}[p(\boldsymbol{\gamma}'_1, \boldsymbol{\gamma}_2)]$. Let $p_1 = E_{\boldsymbol{\gamma}_2}[p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)]$, and $p'_1 = E_{\boldsymbol{\gamma}_2}[p(\boldsymbol{\gamma}'_1, \boldsymbol{\gamma}_2)]$. Then, $p_1 < p'_1$. Without loss of generality, assume that $\boldsymbol{\gamma}_1 = (1, ..., N)$. We next show that recipient 1 with true ranking $\boldsymbol{\gamma}_1$ will have an incentive to misreport if her valuations of different products are sufficiently close to each other. In particular, suppose recipient 1's true valuations are $v_{1,j} = 1 - j\eta, j = 1, ..., N$, where η is a small number such that $p_1 < p'_1(1 - N\eta)$. If recipient 1 truthfully reports her ranking $\boldsymbol{\gamma}_1$, then her expected payoff at $t = 0$ is at most $p_1 K$. If she misreports her ranking as $\boldsymbol{\gamma}'_1$, then her expected payoff at $t = 0$ is at least $p'_1(1 - N\eta)K$, which is strictly larger than $p_1 K$ by construction. Therefore, the payoff vector of truth telling is lexicographically dominated by that of misreporting, which contradicts truthfulness.

Finally, it remains to show that $\tilde{p}_1 + \tilde{p}_2 = 1$. Note that $E_{\boldsymbol{\gamma}_2}[p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)] = \tilde{p}_1, \forall \boldsymbol{\gamma}_1 \in \Gamma$ implies that if recipient 2 plays a mixed strategy of ranking reporting that is consistent with recipient 1's belief on $\boldsymbol{\gamma}_2$, then no matter what recipient 1 reports, the probability for recipient 1 to be selected is always $\tilde{p}_1$. Then, the probability for recipient 1 to be selected must be $\tilde{p}_1$ if recipient 1 plays a mixed strategy that is consistent with recipient 2's belief on γ_1 while recipient 2 plays a mixed strategy that is consistent with recipient 1's belief on γ_2. Similarly, $E_{\gamma_1}[1 - p(\gamma_1, \gamma_2)] = \tilde{p}_2, \forall \gamma_2 \in \Gamma$ implies that the probability for recipient 1 to be selected is $1 - \tilde{p}_2$ if recipient 1 plays a mixed strategy that is consistent with recipient 2's belief on γ_1 while recipient 2 plays a mixed strategy that is consistent with recipient 1's belief on γ_2. Therefore, we must have $\tilde{p}_1 = 1 - \tilde{p}_2$. $\square$

Proof of Proposition 2. We first prove the "if" part of the proposition, i.e., given a score function $s_i = g(\gamma_i, \boldsymbol{\mu}), i = 1, 2$, the associated mechanism (defined in Equation 6) is both symmetric and acyclic. First, the mechanism is symmetric because the two recipients share the same score function, i.e., the score of each recipient only depends on the recipient's reported rankings, but not the specific index of the recipient. Second, for any fixed inventory vector $\boldsymbol{\mu}$, if $p(\gamma_1, \gamma_2, \boldsymbol{\mu}) \geq \frac{1}{2}$ and $p(\gamma_2, \gamma_3, \boldsymbol{\mu}) \geq \frac{1}{2}$, then we have $g(\gamma_1, \boldsymbol{\mu}) \geq g(\gamma_2, \boldsymbol{\mu})$ and $g(\gamma_2, \boldsymbol{\mu}) \geq g(\gamma_3, \boldsymbol{\mu})$. Hence, we have $g(\gamma_1, \boldsymbol{\mu}) \geq g(\gamma_3, \boldsymbol{\mu})$, which implies that $p(\gamma_1, \gamma_3, \boldsymbol{\mu}) \geq \frac{1}{2}$ (clearly, the last inequality holds as equality if and only if the first two inequalities in the condition are both equalities). Therefore, the mechanism is also acyclic.

We now prove the "only if" part, i.e., if a mechanism $p(\gamma_1, \gamma_2, \boldsymbol{\mu})$ is symmetric and acyclic, then it can be characterized by a score function $s_i = g(\gamma_i, \boldsymbol{\mu}), i = 1, 2$. First note that if a mechanism is symmetric and acyclic, then for any given $\boldsymbol{\mu}$, we are able to define a *total order* on the set of rankings Γ. In particular, we define a relation $\leq_{\boldsymbol{\mu}}$ as follows: For any pair of rankings $\gamma_1, \gamma_2 \in \Gamma$, we say $\gamma_1 \leq_{\boldsymbol{\mu}} \gamma_2$ if $p(\gamma_1, \gamma_2, \boldsymbol{\mu}) \leq \frac{1}{2}$, and $\gamma_1 =_{\boldsymbol{\mu}} \gamma_2$ if $p(\gamma_1, \gamma_2, \boldsymbol{\mu}) = \frac{1}{2}$. Then, it is not difficult to check that $\leq_{\boldsymbol{\mu}}$ satisfies the following four properties of a total order: reflectivity (i.e., $\gamma_1 \leq_{\boldsymbol{\mu}} \gamma_1, \forall \gamma_1 \in \Gamma$), antisymmetry (i.e., $\gamma_1 \leq_{\boldsymbol{\mu}} \gamma_2$ and $\gamma_2 \leq_{\boldsymbol{\mu}} \gamma_1$ implies $\gamma_1 =_{\boldsymbol{\mu}} \gamma_2$), transitivity (i.e., $\gamma_1 \leq_{\boldsymbol{\mu}} \gamma_2$ and $\gamma_2 \leq_{\boldsymbol{\mu}} \gamma_3$ implies $\gamma_1 \leq_{\boldsymbol{\mu}} \gamma_3$), and comparability (i.e., for any $\gamma_1, \gamma_2 \in \Gamma$, either $\gamma_1 \leq_{\boldsymbol{\mu}} \gamma_2$ or $\gamma_2 \leq_{\boldsymbol{\mu}} \gamma_1$). Therefore, by definition, $\leq_{\boldsymbol{\mu}}$ defines a total order on Γ. Let $<_{\boldsymbol{\mu}}$ be the associated strict total order, for which we say $\gamma_1 <_{\boldsymbol{\mu}} \gamma_2$ if and only if $\gamma_1 \leq_{\boldsymbol{\mu}} \gamma_2$ and $\gamma_1 \neq_{\boldsymbol{\mu}} \gamma_2$. Then, we are able to assign a score $g(\gamma_i, \boldsymbol{\mu})$ to each ranking vector γ_i such that for any pair of rankings $\gamma_1, \gamma_2 \in \Gamma$, $g(\gamma_1, \boldsymbol{\mu}) < g(\gamma_2, \boldsymbol{\mu})$ if $\gamma_1 <_{\boldsymbol{\mu}} \gamma_2$, and $g(\gamma_1, \boldsymbol{\mu}) = g(\gamma_2, \boldsymbol{\mu})$ if $\gamma_1 =_{\boldsymbol{\mu}} \gamma_2$.

Then, by construction, we have:

$$p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu}) = \begin{cases} 1 & \text{if } g(\boldsymbol{\gamma}_1, \boldsymbol{\mu}) > g(\boldsymbol{\gamma}_2, \boldsymbol{\mu}), \\ 0 & \text{if } g(\boldsymbol{\gamma}_1, \boldsymbol{\mu}) < g(\boldsymbol{\gamma}_2, \boldsymbol{\mu}), \\ \frac{1}{2}, & \text{if } g(\boldsymbol{\gamma}_1, \boldsymbol{\mu}) = g(\boldsymbol{\gamma}_2, \boldsymbol{\mu}), \end{cases}$$

which completes the proof. □

Proof of Proposition 3. First note that if $\mu_1 = ... = \mu_N$, then the score of each recipient i is equal to $s_i = \sum_{j=1}^{N} g_j \mu_{\rho_{i,j}} = \sum_{j=1}^{N} g_j \mu_1$. In this case, the score of the two recipients are always equal no matter what they report. Therefore, it is sufficient to consider the case where the inventory levels of some products are different. Suppose $\mu_1 < \mu_2$ and that there exist $j, k \in \{1, ..., N\}$ such that $g_j > g_k$. Then, reporting different rankings can lead to different scores. For example, let $\boldsymbol{\gamma}_i$ and $\boldsymbol{\gamma}_i'$ be two ranking vectors such that $\gamma_{i,1} = j, \gamma_{i,2} = k, \gamma_{i,1}' = k, \gamma_{i,2}' = j$ and $\gamma_{i,l} = \gamma_{i,l}', l = 3, ..., N$. Then, reporting $\boldsymbol{\gamma}_i'$ leads to $g_k\mu_1 + g_j\mu_2 - (g_j\mu_1 + g_k\mu_2) = (g_j - g_k)(\mu_2 - \mu_1) > 0$ more points than reporting $\boldsymbol{\gamma}_i$. Therefore, under such a mechanism, recipients whose true ranking is $\boldsymbol{\gamma}_i$ may have an incentive to misreport to obtain a higher score and hence a higher probability of being selected (recall that $\phi_{i,-i}(\boldsymbol{v}_{-i}) > 0, \forall \boldsymbol{v}_{-i} \in V$, hence a strictly higher score implies a strictly higher probability of being selected). As shown in Proposition 1, a strictly higher probability of being selected implies a strictly higher expected payoff Π_i^0 if recipient i's valuations of different products are sufficiently close to each other. By definition of truthfulness, such a recipient i has an incentive to misreport. Therefore, we must have $g_1 = ... = g_N$. □

Proof of Theorem 1. Proposition 3 says that the only truthful additive linear score function for case I is such that $g_1 = ... = g_N$. When $g_1 = ... = g_N$, the scores of the two recipients are always equal under any inventory level. Clearly, such a score function is also truthful for case NI. Hence, the set of truthful additive linear score functions for case NI is at least as large as that for case I. Therefore, the optimal truthful additive linear score function for case NI performs no worse than that for case I, i.e., $\pi_{NI} \geq \pi_I$. □

Proof of Proposition 4. We first prove the "if" part of the proposition. In particular, we show that for case NI, if $g_1 \geq g_2 = ... = g_N$, or $g_1 = ... = g_{N-1} \geq g_N$, then the score function $s_i = \sum_{j=1}^N g_j \mu_{\rho_{i,j}}, i = 1,2$ is truthful regardless of recipients' belief on each other's rankings (note that this is in fact more general than the "if" part of the proposition). First consider $g_1 \geq g_2 = ... = g_N$. In this case, $s_i = (g_1 - g_2)\mu_{\rho_{i,1}} + g_2 \sum_{j=1}^N \mu_{\rho_{i,j}}$, hence the mechanism is such that the recipient whose reported rank-1 product has a higher inventory level is selected. Then, it is sufficient to show that recipients do not have an incentive to misreport their rank-1 product (because fixing the reported rank-1 product, misreporting the relative rankings of other products will not change the probability for the recipient to be selected and may decrease the container value when she is selected). Let $\boldsymbol{\gamma}_i$ be recipient i's true ranking, and without loss of generality, assume $\gamma_{i,1} = 1$, i.e., recipient i's true rank-1 product is product 1. Consider the following two cases:

Case 1: Recipient $-i$'s reported rank-1 product is product 1. Suppose recipient i truthfully reports her rank-1 product. Then, the scores of the two recipients are the same and recipient i is selected with probability $\frac{1}{2}$ under any inventory levels. Suppose recipient i misreports her rank-1 product as some product $k \neq 1$. Then, recipient i is selected with probability one if $\mu_1 < \mu_k$, with probability zero if $\mu_1 > \mu_k$, and with probability $\frac{1}{2}$ if $\mu_1 = \mu_k$. Let $\boldsymbol{\gamma}'_i$ be any ranking vector such that $\gamma'_{i,1} = k$. Then, the difference between the expected payoff at $t = 0$ by reporting $\boldsymbol{\gamma}'_i$ and $\boldsymbol{\gamma}_i$ is as follows:

$$
\begin{aligned}
&E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_{-i}}[\sum_{j=1}^N v_{i,j} x^0_{i,j}(\boldsymbol{\gamma}'_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu}) - \sum_{j=1}^N v_{i,j} x^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})]] \\
=&P(\mu_1 < \mu_k) E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_{-i}}[\sum_{j=1}^N v_{i,j} x^0_{i,j}(\boldsymbol{\gamma}'_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu}) - \sum_{j=1}^N v_{i,j} x^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})]|\mu_1 < \mu_k] \\
&+ P(\mu_1 > \mu_k) E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_{-i}}[\sum_{j=1}^N v_{i,j} x^0_{i,j}(\boldsymbol{\gamma}'_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu}) - \sum_{j=1}^N v_{i,j} x^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})]|\mu_1 > \mu_k] \\
&+ P(\mu_1 = \mu_k) E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_{-i}}[\sum_{j=1}^N v_{i,j} x^0_{i,j}(\boldsymbol{\gamma}'_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu}) - \sum_{j=1}^N v_{i,j} x^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})]|\mu_1 = \mu_k] \\
=&P(\mu_1 < \mu_k) E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_{-i}}[1 \times \sum_{j=1}^N v_{i,j} y^0_{i,j}(\boldsymbol{\gamma}'_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu}) - \frac{1}{2} \times \sum_{j=1}^N v_{i,j} y^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})]|\mu_1 < \mu_k] \\
&+ P(\mu_1 > \mu_k) E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_{-i}}[0 \times \sum_{j=1}^N v_{i,j} y^0_{i,j}(\boldsymbol{\gamma}'_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu}) - \frac{1}{2} \times \sum_{j=1}^N v_{i,j} y^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})]|\mu_1 > \mu_k] \\
&+ P(\mu_1 = \mu_k) E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_{-i}}[\frac{1}{2} \times \sum_{j=1}^N v_{i,j} y^0_{i,j}(\boldsymbol{\gamma}'_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu}) - \frac{1}{2} \times \sum_{j=1}^N v_{i,j} y^0_{i,j}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})]|\mu_1 = \mu_k]
\end{aligned}
$$

$$
\begin{aligned}
&\leq P(\mu_1 < \mu_k) E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_{-i}}[\frac{1}{2} \times \sum_{j=1}^{N} v_{i,j} y_{i,j}^0(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})] | \mu_1 < \mu_k] \\
&\quad + P(\mu_1 > \mu_k) E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_{-i}}[-\frac{1}{2} \times \sum_{j=1}^{N} v_{i,j} y_{i,j}^0(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})] | \mu_1 > \mu_k] \\
&< 0,
\end{aligned}
$$

where the first inequality holds because $\sum_{j=1}^{N} v_{i,j} y_{i,j}^0(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu}) \geq \sum_{j=1}^{N} v_{i,j} y_{i,j}^0(\boldsymbol{\gamma}_i', \boldsymbol{\gamma}_{-i}, \boldsymbol{\mu})$ under any values of μ_1 and μ_k, i.e., given that recipient i is served at $t = 0$, truth-reporting maximizes the container value regardless of the inventory availability; the second inequality holds because i) $P(\mu_1 < \mu_k) = P(\mu_1 > \mu_k)$ due to symmetry, and ii) fixing $\mu_1 + \mu_k$, the total value of recipient i's best bundle (defined by Equation 1 in the paper) is strictly higher when $\mu_1 > \mu_k$ than when $\mu_1 < \mu_k$. Therefore, misreporting leads to a strictly smaller expected payoff at $t = 0$, hence recipients do not have an incentive to misreport.

Case 2: Recipient $-i$'s reported rank-1 product is some product $k \neq 1$. Suppose recipient i truthfully reports her rank-1 product. Then, recipient i is selected with probability one if $\mu_1 > \mu_k$, with probability zero if $\mu_1 < \mu_k$, and with probability $\frac{1}{2}$ if $\mu_1 = \mu_k$. Suppose recipient i misreports her rank-1 product as product k. Then, recipient i is selected with probability $\frac{1}{2}$ under any inventory levels. Then, following a similar argument as above, we know that misreporting leads to a strictly smaller expected payoff at $t = 0$ for recipient i. Hence, recipient i does not have an incentive to misreport her rank-1 product as product k. Similarly, recipient i does not have an incentive to misreport her rank-1 product as any product $l \neq 1, k$, which increases her probability of being selected when $\mu_1 < \mu_l$ but decreases her probability of being selected when $\mu_1 > \mu_l$.

Combining Cases 1 and 2, we know that recipients do not have an incentive to misreport when $g_1 \geq g_2 = ... = g_N$.

When $g_1 = ... = g_{N-1} \geq g_N$, the score of each recipient i is $s_i = g_1 \sum_{j=1}^{N} \mu_{\rho_{i,j}} - (g_1 - g_N)\mu_{\rho_{i,N}}$, hence the mechanism is such that the recipient whose reported rank-N product has a lower inventory is selected. Similar to before, it is sufficient to show that recipients do not have an incentive to misreport their rank-N product, and the proof follows the same manner as for $g_1 \geq g_2 = ... = g_N$.

We now prove the "only if" part, i.e., for case NI-C, if a score function $s_i = \sum_{j=1}^N g_j \mu_{\rho_{i,j}}, i = 1, 2$ is truthful, then either $g_1 \geq g_2 = ... = g_N$ or $g_1 = ... = g_{N-1} \geq g_N$. We prove this part in the following three steps:

Step 1: We show that if a score function $s_i = \sum_{j=1}^N g_j \mu_{\rho_{i,j}}, i = 1, 2$ is truthful, then we must have $g_1 \geq ... \geq g_N$ regardless of recipients' belief on each other's rankings. Suppose this is not the case, i.e., there exist $1 \leq k < l \leq N$ such that $g_k < g_l$. Without loss of generality, assume that recipient 1 believes that recipient 2's true ranking is $\boldsymbol{\gamma}_2 = (1, ..., N)$ with probability $1 - \theta > 0$ (note that in Step 1, we only need $\theta < 1$). We next construct a valuation $\boldsymbol{v}_1 \in V$ and a belief $\psi_1 \in \Psi$ so that recipient 1 has an incentive to misreport. In particular, let $\eta_1, \eta_2 > 0$ be two small numbers. We show that recipient 1 has an incentive to reverse the rankings of products k and l if i) her true valuation is such that $\boldsymbol{\gamma}_1 = (1, ..., N)$ and $v_{1,1} \leq 1, v_{1,k} \geq 1 - \eta_1, v_{1,l} \leq \eta_1$; and ii) she believes that with probability $1 - \eta_2$, one of the N products has an inventory level of one while all other products have zero inventory, i.e., $\sum_{j=1}^N \mu_j = 1$ (note that here we implicitly assume $K = 1$; if $K > 1$, then we can modify the construction by scaling up the inventory levels of different products proportionally).

Conditioned on $\sum_{j=1}^N \mu_j = 1$ (with probability $1 - \eta_2$), reversing the rankings of products k and l affects recipient 1's probability of being selected and payoff at $t = 0$ only when $\mu_k = 1$ or $\mu_l = 1$ (with conditional probability $\frac{1}{N}$ each due to symmetry). Suppose recipient 2's ranking is $\boldsymbol{\gamma}_2 = (1, ..., N)$ (with probability $1 - \theta$). If recipient 1 truthfully reports her ranking, then she is selected with probability $\frac{1}{2}$ under any inventory levels. If recipient 1 reverses the rankings of products k and l while truthfully reporting the rankings of all other products, then she is selected with probability one when $\mu_k = 1$ (because $s_1 = g_l > g_k = s_2$), and with probability zero when $\mu_l = 1$ (because $s_1 = g_k < g_l = s_2$). Suppose recipient 2's ranking $\boldsymbol{\gamma}_2 \neq (1, ..., N)$ (with probability θ). Then similarly, reversing the rankings of products k and l will only increase recipient 1's probability of being selected unless $\mu_l = 1$. Therefore, by reversing the rankings of products k

and l, the expected payoff at $t = 0$ will increase by:

$$\begin{aligned}
&E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_2}[\sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2, \boldsymbol{\mu}) - \sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})]] \\
=&(1-\eta_2)(1-\theta)E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_2}[\sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2, \boldsymbol{\mu}) - \sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})|\boldsymbol{\gamma}_2 = (1, ..., N)]|\sum_{j=1}^{N}\mu_j = 1] \\
&+ (1-\eta_2)\theta E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_2}[\sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2, \boldsymbol{\mu}) - \sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})|\boldsymbol{\gamma}_2 \neq (1, ..., N)]|\sum_{j=1}^{N}\mu_j = 1] \\
&+ \eta_2 E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_2}[\sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2, \boldsymbol{\mu}) - \sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})]|\sum_{j=1}^{N}\mu_j \neq 1].
\end{aligned}$$

As discussed above, when $\sum_{j=1}^{N}\mu_j = 1$, reversing the rankings of products k and l will affect recipient 1's payoff at $t = 0$ only when $\mu_k = 1$ or $\mu_l = 1$. Therefore, we have:

$$\begin{aligned}
&E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_2}[\sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2, \boldsymbol{\mu}) - \sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})|\boldsymbol{\gamma}_2 = (1, ..., N)]|\sum_{j=1}^{N}\mu_j = 1] \\
=&\frac{1}{N}E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_2}[\sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2, \boldsymbol{\mu}) - \sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})|\boldsymbol{\gamma}_2 = (1, ..., N)]|\mu_k = 1] \\
&+ \frac{1}{N}E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_2}[\sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2, \boldsymbol{\mu}) - \sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})|\boldsymbol{\gamma}_2 = (1, ..., N)]|\mu_l = 1] \\
=&\frac{1}{N} \times (1 \times v_{1,k} - \frac{1}{2} \times v_{1,k}) + \frac{1}{N} \times (0 \times v_{1,l} - \frac{1}{2} \times v_{1,l}) \\
=&\frac{1}{N} \times \frac{1}{2} \times (v_{1,k} - v_{1,l}).
\end{aligned}$$

Then:

$$
\begin{aligned}
&E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_2}[\sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2, \boldsymbol{\mu}) - \sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})]] \\
=&(1-\eta_2)(1-\theta) \times \frac{1}{N} \times \frac{1}{2} \times (v_{1,k} - v_{1,l}) \\
&+ (1-\eta_2)\theta E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_2}[\sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2, \boldsymbol{\mu}) - \sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu}) | \boldsymbol{\gamma}_2 \neq (1, ..., N)]| \sum_{j=1}^{N} \mu_j = 1] \\
&+ \eta_2 E_{\boldsymbol{\mu}}[E_{\boldsymbol{\gamma}_2}[\sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2, \boldsymbol{\mu}) - \sum_{j=1}^{N} v_{1,j}x_{1,j}^0(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})]| \sum_{j=1}^{N} \mu_j \neq 1] \\
\geq&(1-\eta_2)(1-\theta) \times \frac{1}{N} \times \frac{1}{2} \times (v_{1,k} - v_{1,l}) + (1-\eta_2)\theta \times \frac{1}{N} \times (-v_{1,l}) + \eta_2 \times (-1) \\
\geq&(1-\eta_2)(1-\theta) \times \frac{1}{N} \times \frac{1}{2} \times (1-\eta_1-\eta_1) - (1-\eta_2)\theta \times \frac{1}{N} \times \eta_1 - \eta_2 \\
>&0,
\end{aligned}
$$

where the first inequality holds because when $\sum_{j=1}^{N} \mu_j = 1$, the expected payoff loss by reversing the rankings of products k and l is upper bounded by $\frac{1}{N} v_{1,l}$ (since there will be payoff loss only when $\mu_l = 1$), and when $\sum_{j=1}^{N} \mu_j \neq 1$, the payoff loss is upper bounded by 1 (recall that by construction, $v_{1,1} \leq 1$ and $K = 1$); the second inequality holds because $v_{1,k} \geq 1 - \eta_1$ and $v_{1,l} \leq \eta_1$; and the last inequality holds as long as η_1, η_2 are sufficiently small and $1 - \theta > 0$.

Hence, reversing the rankings of products k and l strictly increases the expected payoff at $t = 0$, which contradicts truthfulness. Therefore, we have $g_1 \geq ... \geq g_N$.

Step 2: Without loss of generality, we assume that $1 = g_1 \geq ... \geq g_N = 0$. We now show that to ensure truthfulness for case NI-C, there must exist $k \in \{1, ..., N-1\}$ such that $g_1 = ... = g_k = 1$ and $g_{k+1} = ... = g_N = 0$. There is nothing to prove for $N \leq 2$. Consider $N \geq 3$. Suppose this is not the case, i.e., there exists $k \in \{2, ..., N-1\}$ such that $0 < g_k < 1$. Assume that recipient 1 believes that recipient 2's true ranking is $\boldsymbol{\gamma}_2 = (1, ..., N)$ with probability $1 - \theta$, where $\theta < \frac{1}{3}$. We next show that recipient 1 may have an incentive to misreport her rankings of products 1 and N as $\gamma_{1,1}' = 1$ and $\gamma_{1,N}' = k$ if her true ranking is such that $\gamma_{1,1} = k$ and $\gamma_{1,N} = 1$ (i.e., to improve the ranking of product 1 from k to 1 and lower the ranking of product N from 1 to k), and she believes that with a large probability $1 - \eta$, one of the N products has an inventory level of one while all other products have zero inventory.

Conditioned on that only one product has an inventory level of one while all other products have zero inventory, misreporting affects recipient 1's probability of being selected only when $\mu_1 = 1$ or $\mu_N = 1$ (with conditional probability $\frac{1}{N}$ each). Suppose recipient 2's ranking is $\boldsymbol{\gamma}_2 = (1, ..., N)$. If recipient 1 truthfully reports her ranking, then she is selected with probability zero when $\mu_1 = 1$ (because $s_1 = g_k < 1 = s_2$), and with probability one when $\mu_N = 1$ (because $s_1 = 1 > 0 = s_2$). If recipient 1 misreports her rankings of products 1 and N as $\gamma'_{1,1} = 1$ and $\gamma'_{1,N} = k$, then she is selected with probability $\frac{1}{2}$ when $\mu_1 = 1$ (because $s_1 = 1 = s_2$), and with probability one when $\mu_N = 1$ (because $s_1 = g_k > 0 = s_2$). Suppose recipient 2's ranking $\boldsymbol{\gamma}_2 \neq (1, ..., N)$. Then similarly, misreporting will only increase recipient 1's probability of being selected unless $\mu_N = 1$. Therefore, by misreporting the rankings of products 1 and N, the probability for recipient 1 to be selected will increase by at least:

$$(1-\eta)(1-\theta) \times \frac{1}{N} \times \frac{1}{2} - (1-\eta)\theta \times \frac{1}{N} - \eta \times 1,$$

which is strictly positive for sufficiently small η since $\theta < \frac{1}{3}$. That is, in this case, misreporting strictly increases the expected payoff at $t = 0$ if recipient 1's valuations for different products are sufficiently close to each other (Proposition 1), which contradicts truthfulness.

Step 3: We show that for case NI-C, the only truthful mechanisms are such that $g_1 \geq g_2 = ... = g_N$ or $g_1 = ... = g_{N-1} \geq g_N$. Given that there exists $k = 1, ..., N-1$ such that $g_1 = ... = g_k = 1$ and $g_{k+1} = ... = g_N = 0$ (i.e., Step 2), there is nothing to prove for $N \leq 3$. Consider $N = 4$. Suppose $g_1 = g_2 = 1$ and $g_3 = g_4 = 0$. Without loss of generality, assume that recipient 1 believes that recipient 2's true ranking is $\boldsymbol{\gamma}_2 = (1, 2, 3, 4)$ with probability $1-\theta$, where $\theta < \frac{1}{3}$. Let $\eta_1, \eta_2 > 0$ be two small numbers. We now show that recipient 1 has an incentive to misreport her ranking as $\boldsymbol{\gamma}'_1 = (2, 4, 1, 3)$ if her true ranking is $\boldsymbol{\gamma}_1 = (3, 4, 1, 2)$, true valuation is $\boldsymbol{v}_1 = (\eta_1, 0, 1, 2\eta_1)$, and she believes that with probability $1 - \eta_2$, two of the four products have an inventory level of one while the other two products have zero inventory. There are six possibilities (with conditional probability $\frac{1}{6}$ each):

Inventory levels $\boldsymbol{\mu}$	Score of recipient 1 (truth-reporting)	Score of recipient 1 (misreporting)	Score of recipient 2 (when $\boldsymbol{\gamma}_2 = (1, 2, 3, 4)$)
(1,1,0,0)	0	1	2
(1,0,1,0)	1	2	1
(1,0,0,1)	1	1	1
(0,1,1,0)	1	1	1
(0,1,0,1)	1	0	1
(0,0,1,1)	2	1	0

Suppose recipient 2's ranking is $\boldsymbol{\gamma}_2 = (1, 2, 3, 4)$. Then misreporting affects recipient 1's probability of being selected only when $\boldsymbol{\mu} = (1, 0, 1, 0)$ or $\boldsymbol{\mu} = (0, 1, 0, 1)$. In particular, if recipient 1 truthfully reports her ranking, then she is selected with probability $\frac{1}{2}$ under both cases. If recipient 1 misreports her ranking as $\boldsymbol{\gamma}_1' = (2, 4, 1, 3)$, then she is selected with probability one when $\boldsymbol{\mu} = (1, 0, 1, 0)$, and with probability zero when $\boldsymbol{\mu} = (0, 1, 0, 1)$. Suppose recipient 2's ranking $\boldsymbol{\gamma}_2 \neq (1, 2, 3, 4)$. Then similarly, misreporting can only increase recipient 1's probability of being selected unless $\boldsymbol{\mu} = (0, 1, 0, 1)$ or $\boldsymbol{\mu} = (0, 0, 1, 1)$. Therefore, by misreporting, recipient 1's expected payoff at $t = 0$ will increase by at least:

$$(1 - \eta_2)(1 - \theta) \times \frac{1}{6} \times \frac{1}{2} \times (1 + \eta_1 - 2\eta_1) - (1 - \eta_2)\theta \times \frac{1}{6} \times (2\eta_1 + 1 + 2\eta_1) - \eta_2 \times 2,$$

which is strictly positive for sufficiently small η_1, η_2 since $\theta < \frac{1}{3}$. That is, in this case, misreporting strictly increases the expected payoff at $t = 0$, which contradicts truthfulness.

For the general case where $N > 4$, the proof follows from a similar construction: Assume that recipient 1 believes that recipient 2's true ranking is $\boldsymbol{\gamma}_2 = (1, ..., N)$ with probability $1 - \theta$. Suppose $g_1 = g_2 = 1$ and $g_{N-1} = g_N = 0$. Then, we can show that recipient 1 has an incentive to misreport her ranking as $\boldsymbol{\gamma}_1' = (2, N, 3, ..., N-2, 1, N-1)$ if her true ranking is $\boldsymbol{\gamma}_1 = (N-1, N, 3, ..., N-2, 1, 2)$, true valuation is $\boldsymbol{v}_1 = (\eta_1, 0, (N-3)\eta_1, ..., 2\eta_1, 1, (N-2)\eta_1)$, and she believes that with probability $1 - \eta_2$, two of the N products have an inventory level of one while other products have zero inventory. Therefore, any mechanism with $g_1 = g_2 = 1$ and $g_{N-1} = g_N = 0$ is not truthful, which completes the proof. □

Proof of Proposition 5. The "only if" part of the proposition follows directly from Step 1 of the proof for the "only if" part of Proposition 4. Therefore, it is sufficient to prove the "if" part, i.e., for case NI-NC, if $g_1 \geq ... \geq g_N$, then the score function $s_i = \sum_{j=1}^{N} g_j \mu_{\rho_{i,j}}, i = 1, 2$ is truthful. Recall that for case NI-NC, recipients' beliefs on inventory levels and competitor's rankings are both symmetric in products. Then, for each recipient, the expected amount of each product she receives is solely determined by her reported ranking of that product (not the specific index of that product). For each recipient i, let $z_{i,j}^0$ denote the expected amount of her reported rank-j product she receives at $t = 0$, where $i = 1, 2; j = 1, ..., N$. Then, it is sufficient to show that if $g_1 \geq ... \geq g_N$, then $z_{i,j}^0 > z_{i,k}^0$ for $1 \leq j < k \leq N$ (if this is true, then misreporting leads to a strictly smaller expected payoff at $t = 0$).

Without loss of generality, assume that recipient i's reported ranking is $\boldsymbol{\gamma}_i = (1, ..., N)$. Fixing the inventory levels of all other products that are not j or k, consider two symmetric cases with equal probability: (i) $\mu_j = a, \mu_k = b$; and (ii) $\mu_j = b, \mu_k = a$, where $0 \leq b \leq a \leq K$. Let $p_a, p_b > 0$ be the probabilities that recipient i will be selected in cases (i) and (ii), respectively. The probabilities are strictly positive because recipient $-i$ can have any possible ranking $\boldsymbol{\gamma}_{-i} \in \Gamma$. Further, we have $p_a \geq p_b$ because recipient i has a higher score in case (i) than in case (ii) (i.e., if recipient i is selected in case (ii), then she must also be selected in case (i)). Therefore, with probability $p_b > 0$, recipient i is selected in both cases (i) and (ii); with probability $p_a - p_b \geq 0$, recipient i is selected in case (i) but not in case (ii); and with probability $1 - p_a \geq 0$, recipient i is selected in neither case (i) or (ii).

To prove $z_{i,j}^0 > z_{i,k}^0$, it is sufficient to show that the inequality holds strictly conditional on that recipient i is selected in both cases (i) and (ii). Since the probability for cases (i) and (ii) is equal and recipient i will only receive product k when she has emptied all product j in inventory, clearly we have $z_{i,j}^0 \geq z_{i,k}^0$. Further, the inequality is strict because $\psi_i(\boldsymbol{\mu}) > 0, \forall \boldsymbol{\mu} \in M$, implying a positive probability that recipient i needs some product j but not all of products j and k to fill her container. $\square$

Proof of Theorem 2. The proof of Theorem 2 directly follows from Propositions 4 and 5. In particular, Propositions 4 and 5 together imply that the set of truthful additive linear score functions for case NI-NC is at least as large as that for case NI-C. Therefore, the optimal truthful additive linear score functions for case NI-NC performs no worse than that for case NI-C, i.e., $\pi_{NI\text{-}NC} \geq \pi_{NI\text{-}C}$. $\square$

Proof of Proposition 6. First, from the proof for the "if" part of Proposition 4, we know that for the case NI, if $g_1 \geq g_2 = ... = g_N$, or $g_1 = ... = g_{N-1} \geq g_N$, then $s_i = \sum_{j=1}^{N} g_j \mu_{\rho_{i,j}}, i = 1, 2$ is truthful regardless of recipients' belief on each other's rankings. Then, for any given recipients' beliefs $\phi_{i,-i}, i = 1, 2$, the set of truthful mechanisms is at least as large as that for case NI-C. Therefore, $\pi_{NI\text{-}C} \leq \pi_{NI}(\phi_{1,2}, \phi_{2,1})$. Second, from Step 1 of the proof for the "only if" part of Proposition 4, we know that $g_1 \geq ... \geq g_N$ is a necessary condition to ensure truthfulness regardless of recipients' belief on each other's rankings. Then, for any given recipients' beliefs $\phi_{i,-i}, i = 1, 2$, the set of truthful mechanisms is at most as large as that for case NI-NC. Therefore, $\pi_{NI\text{-}NC} \geq \pi_{NI}(\phi_{1,2}, \phi_{2,1})$. $\square$

Proof of Proposition 7. We prove this proposition in the following two steps.

Step 1: We show that to ensure truthfulness, $\sum_{j=1}^{N} h(\tilde{v}_{i,j})$ must be a constant C. Suppose this is not the case, i.e., there exist $\boldsymbol{v}_i$ and $\boldsymbol{v}_i'$ such that $\sum_{j=1}^{N} h(\tilde{v}_{i,j}) < \sum_{j=1}^{N} h(\tilde{v}_{i,j}')$. Without loss of generality, assume that the associated rankings of $\boldsymbol{v}_i$ and $\boldsymbol{v}_i'$ are the same. Let η be a small number. We now show that recipient i has an incentive to misreport her valuation as $\boldsymbol{v}_i'$ if her true valuation is $\boldsymbol{v}_i$ and she believes that with probability $1 - \eta$, the inventory levels of all products are equal, i.e., $\mu_1 = ... = \mu_N$.

When $\mu_1 = ... = \mu_N$, the score for recipient i is $s_i = (\sum_{j=1}^{N} g_j + \sum_{j=1}^{N} h(\tilde{v}_{i,j}))\mu_1$, in which case reporting $\boldsymbol{v}_i'$ leads to a strictly higher score and hence a strictly larger probability of being selected than reporting $\boldsymbol{v}_i$. Let $q > 0$ denote the difference between the probabilities for recipient i to be selected by reporting $\boldsymbol{v}_i'$ and $\boldsymbol{v}_i$, respectively. Then, by misreporting, recipient i's expected payoff at $t = 0$ is at least $(1 - \eta)qv_{i,\rho_{i,1}}\mu_1 - \eta v_{i,\rho_{i,1}}K$ more than that of truth-reporting. Clearly, this difference is strictly positive for sufficiently small η. Hence, recipient i has an incentive to misreport, which contradicts truthfulness.

Step 2: We show that to ensure truthfulness, we must have $\sum_{j=1}^{N} h(\tilde{v}_{i,j}) = C = 0$, implying $h(\tilde{v}_{i,j}) = 0, \forall \tilde{v}_{i,j}$. Suppose this is not the case. Assume that $C = 1$. Consider the following two cases:

Case 1: $N = 2$. In this case, by reporting $\boldsymbol{v}_i$ with associated ranking $\boldsymbol{\gamma}_i = (1, 2)$, recipient i

gets $g_1 + h(\tilde{v}_{i,1})$ points for each unit of product 1 in inventory, and $g_2 + h(\tilde{v}_{i,2})$ points for each unit of product 2 in inventory. Since $\sum_{j=1}^{2}(g_j + h(\tilde{v}_{i,j})) = g_1 + g_2 + 1$ is a constant, the decision of reporting different valuations is equivalent to allocating a total of $g_1 + g_2 + 1$ fixed points between the two products (with g_1 and g_2 going to two different products), and the recipient who allocates more points to product 1 is selected if $\mu_1 > \mu_2$, the other recipient is selected if $\mu_1 < \mu_2$, and a random recipient is selected if $\mu_1 = \mu_2$. Then, under symmetric beliefs on inventory levels, the probability for each recipient to be selected is always $\frac{1}{2}$. We now show that any recipient i with true valuations $1 > \tilde{v}_{i,1} > \tilde{v}_{i,2} > 0$ has an incentive to misreport her valuations as $\tilde{v}'_{i,1} = 1, \tilde{v}'_{i,2} = 0$. This is true because allocating more points to product 1 increases recipient i's probability of being selected when $\mu_1 > \mu_2$ (while decreasing the probability of being selected by the same extent when $\mu_1 < \mu_2$), which increases recipient i's expected payoff at $t = 0$. Hence, recipient i has an incentive to misreport.

Case 2: $N > 2$. In this case, we first show that $h(\tilde{v}_{i,j})$ must be a linear function of $\tilde{v}_{i,j}$. Since $\sum_{j=1}^{N} h(\tilde{v}_{i,j}) = 1$ for any valuation vector, for any $s, x \in [0, 1]$ such that $1 - s(1 + x) \geq 0$, we must have $h(s) + h(sx) + h(1 - s(1 + x)) = 1$. Since h is monotonically nondecreasing, it must be differentiable almost everywhere. Then, we have $sh'(sx) - sh'(1 - s(1 + x)) = 0$. Let $s > 0$ and $x = 0$, then $h'(0) = h'(1 - s)$. Hence, the derivative of h must be a constant in $[0, 1]$. Since $h(0) = 0$ and $\sum_{j=1}^{N} h(\tilde{v}_{i,j}) = 1$, we must have $h(\tilde{v}_{i,j}) = \tilde{v}_{i,j}$.

Note that due to symmetry in recipients, there must exist a valuation vector $\boldsymbol{v}_i \in V$ such that by reporting $\boldsymbol{v}_i$, the probability for recipient i to be selected is no larger than $\frac{1}{2}$. Without loss of generality, assume $\boldsymbol{\gamma}_i(\boldsymbol{v}_i) = (1, ..., N)$. Let η be a small number. We now show that recipient i with true valuation $\boldsymbol{v}_i$ has an incentive to misreport her valuation as $v'_{i,1} = 1, v'_{i,j} = (N - j)\eta, j = 2, ..., N$ if she believes that one of the N products have zero inventory while all other products have an inventory level of one with a large probability (for simplicity, we proceed by assuming that this probability is equal to one).

Since both $\sum_{j=1}^{N} g_j$ and $\sum_{j=1}^{N} h(\tilde{v}_{i,j})$ are constants independent of $\boldsymbol{v}_i$, to compare the score of the two recipients, it is sufficient to compare the amount of points each recipient loses due to the associated product not being available. Consider the following two possibilities: If $\mu_1 = 0$ and $\mu_j = 1, \forall j \geq 2$ (with probability $\frac{1}{N}$), then misreporting leads to a loss of $g_1 + 1 - o(\eta)$ points (since

$h(\tilde{v}_{i,j}) = \tilde{v}_{i,j}$) and hence recipient i will only be selected with probability $o(\eta)$. If $\mu_j = 0$ for some $j \geq 2$ and $\mu_k = 1, \forall k \neq j$ (with probability $\frac{1}{N}$ for each j), then misreporting leads to a loss of $g_j + o(\eta)$ points and hence recipient i will be selected with probability at least $\frac{j}{N} - o(\eta)$ (i.e., when recipient $-i$'s ranking of product j is higher than or equal to j). Therefore, the overall probability for recipient i to be selected is at least:

$$\frac{1}{N} \times (\frac{2}{N} + ... + \frac{N}{N}) - o(\eta) = \frac{(N+2)(N-1)}{2N^2} - o(\eta),$$

which is strictly larger than $\frac{1}{2}$ when η is sufficiently small. Further, misreporting decreases recipient i's probability of being selected only when $\mu_1 = 0, \mu_j = 1, \forall j \geq 2$, in which case the value of recipient i's best bundle at $t = 0$ is strictly lower than when $\mu_j = 0$ for some $j \geq 2, \mu_k = 1, \forall k \neq j$. Hence, misreporting leads to a strictly higher expected payoff at $t = 0$, which is a contradiction. □

Proof of Theorem 3. The proof of Theorem 3 directly follows from Propositions 5 and 7. Propositions 5 and 7 together imply that the set of truthful additive linear score functions for case NI-NC in the cardinal case is the same as that in the ordinal case. Therefore, $\pi^C_{NI\text{-}NC} = \pi_{NI\text{-}NC}$. □

C.2 Model Extensions

C.2.1 Multiple Recipients

In practice, there are typically multiple recipients in the system waiting to be served. For example, in the case of MedShare, there are usually 5-10 recipients for whom funding has been secured for a container shipment. In this section, we consider an extension where T recipients are served in T respective periods.

We ask all of the T recipients to report their preference rankings of different products at the beginning of $t = 0$, and the decision is to select a recipient among the remaining $T - t$ recipients at each period $t = 0, ..., T - 2$. We note that when the time horizon is not too large (e.g., in the case of MedShare, a period is about one week, and the horizon length is about five weeks when five recipients are considered), it is reasonable to assume that recipient needs for critical products do not change within the horizon.

Recall that in the main text, we have primarily analyzed additive linear score functions. Such a scoring approach can be easily applied to the multi-recipient setting: At each period t, based on the reported rankings of recipients and the MSRO inventory levels $\mu_j^t, j = 1, ..., N$, each of the remaining $T-t$ recipients are assigned a score $s_i = \sum_{j=1}^{N} g_j \mu_{\rho_{i,j}}^t, i = 1, ..., T-t$, and the recipient with the highest score is selected. If there is a tie among $k \leq T - t$ recipients (i.e., their scores are the same and are the highest among the $T - t$ recipients), then each of the k recipients is selected with probability $\frac{1}{k}$.

Based on the above setup, we show that in the multi-recipient setting, i) the conclusion that there is nonnegative value of eliminating inventory and competitor information continues to hold; ii) when both inventory and competitor information is eliminated, any additive linear score function with $g_1 \geq ... \geq g_N$ remains truthful; and iii) there is no value added from further eliciting recipient valuations. We refer the formal characterization and proof of all these results to our Technical Appendix (available upon request).

Finally, we find that for case NI-C, an additive linear score function with $g_1 = 1, g_2 = ... = g_N = 0$ remains truthful under the following modification: At $t = 0$, after all T recipients report their rankings, we randomly sort the recipients from 1 to T. Then, for recipient 1, we simply set her score as the inventory level of her rank-1 product; for each recipient $i = 2, ..., T$, we define her score as the inventory level of her most preferred product among those that have not been used to define the scores of recipients $1, ..., i - 1$. This way, the score of each recipient is equal to the inventory level of a single product, and the scores of different recipients are defined as the inventory levels of different products.

C.2.2 Asymmetric Score Functions

Recall that in the main text, we studied score functions that are symmetric in recipients, i.e., both recipients share the same score function $s_i = g(\boldsymbol{\gamma}_i, \boldsymbol{\mu}), i = 1, 2$. In this section, we present an extension to consider asymmetric (i.e., recipient-specific) score functions.

In particular, we now consider a general score function $s_i = g_i(\boldsymbol{\gamma}_i, \boldsymbol{\mu}), i = 1, 2$, and extend our characterization of truthful mechanisms for case I (i.e., Proposition 3) to this general setting. More specifically, we show that for case I, a score function $s_i = g_i(\boldsymbol{\gamma}_i, \boldsymbol{\mu}), i = 1, 2$ is truthful if and only if the associated mechanism $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$ defined in Equation 7 is a constant for any $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \Gamma$.

Clearly, if symmetry in recipients is required, i.e., $g_1(\boldsymbol{\gamma}_i, \boldsymbol{\mu}) = g_2(\boldsymbol{\gamma}_i, \boldsymbol{\mu}), \forall \boldsymbol{\gamma}_i \in \Gamma$, then $p(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\mu})$ has to be $\frac{1}{2}, \forall \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \Gamma$. Hence, the above result generalizes Proposition 3 in the following two ways: i) from an additive linear score function to a general score function, and ii) from a symmetric mechanism (i.e., two recipients sharing the same score function) to an asymmetric mechanism.

For case NI, similar to before, we consider additive linear score functions, except that we now allow a base score $s_{0,i}$ that can be different for different recipients: $s_i = s_{0,i} + \sum_{j=1}^{N} g_j \mu_{\rho_{i,j}}, i = 1, 2$, where $g_1 \geq ... \geq g_N$. We show that under such a score function, i) the conclusion that there is nonnegative value of eliminating inventory and competitor information continues to hold; ii) when both inventory and competitor information is eliminated, any additive linear score function with $g_1 \geq ... \geq g_N$ remains truthful; and iii) under additional conditions, there is no value added from eliciting recipient valuations. We refer the formal characterization and proof of all these results to our Technical Appendix.

C.2.3 Bounded Perturbation on Recipients' Beliefs

In the main text, we assumed that recipient beliefs on inventory levels and each other's rankings are both symmetric over products in case NI-NC. In this section, we show that our result is in a way robust against bounded perturbations on recipient beliefs (i.e., when recipients believe that some product is slightly more likely to have a higher inventory than other products, or that the other recipient is slightly more likely to prefer some product than other products). More specifically, we show that under bounded perturbations on recipients' beliefs, any recipient whose valuations of different products are sufficient different does not have an incentive to misreport under our proposed mechanisms.

While we refer the formal characterization and proof of the above result to our Technical Appendix, the key intuition of this result is that when a recipient have asymmetric beliefs on inventory and competitor preference, then she may have an incentive to improve the ranking of some product (e.g., a product that is likely to have a higher inventory level than other products) if there is another product of higher ranking but similar valuation. In contrast, if all products of higher rankings have significantly higher valuations than this product, then misreporting will reduce the expected payoff.

C.2.4 Bayesian Update of Recipients' Belief on MSRO Inventory

Recall that for case NI, we assumed that the MSRO inventory information is private to the MSRO and that recipients have a prior belief on the MSRO inventory levels. Since MSROs with different inventory levels may prefer different mechanisms, theoretically recipients may be able to update their belief and infer the inventory information based on the announced mechanism.

We prove that the above is not a problem under an additive linear score function mechanism that we consider. More specifically, we show that under any set of announced g_j's, recipients' posterior belief on the MSRO inventory remains symmetric in products. The key intuition is that under an additive linear score function, each recipient gets g_j points for each unit of her rank-j product (i.e., product $\rho_{i,j}$) instead of product j, hence recipients are not able to infer the inventory levels of different products when they have a symmetric prior on both the MSRO inventory levels and the other recipient's rankings. We refer the formal characterization and proof of this result to our Technical Appendix.

C.2.5 An Alternative Definition of Truthfulness

In the main text, we defined truthfulness based on a lexicographical dominance, which capture the observation that recipients are typically impatient for critical products can compete for the shipping opportunity. An alternative way to define truthfulness is to consider a discounted payoff model and assume that recipients will truthfully report their ranking or valuation when truth-reporting results in a total discounted payoff at least as large as that of reporting any other ranking or valuation.

We prove that under this alternative definition of truthfulness, i) the conclusion that for case I, the only truthful (symmetric and acyclic) mechanism is random selection among recipients continues to hold; and ii) when recipients have symmetric beliefs on the MSRO inventory levels, the other recipient's ranking, as well as the arrival quantities of different products, an additive linear score function with monotone coefficients (with coefficients of non-top-ranked products being zero) remains truthful. We refer the formal characterization and proof of this result to our Technical Appendix.

C.2.6 Heuristics for Relaxing the "Best Bundle" Assumption

A key feature of MedShare's existing resource allocation model is that each recipient is allowed to pick their own products to fill a container. This practice is intuitive and expected to significantly reduce the mismatch between supply and demand compared with traditional "push" models implemented in this context where products are sent to recipients without explicitly considering recipient needs.

The objective of our paper is to improve upon the above industry best practice by helping MSROs select the ideal recipient to serve at each shipping opportunity based on recipient needs and the MSRO inventory levels; once a recipient is selected, following a similar structure as the existing practice, we assumed that the recipient receives the "best bundle" (based on reported preferences) from the MSRO inventory. While this practice is appealing from implementation perspectives, one way that can potentially further improve the MSRO's value provision capability is to relax the best bundle assumption and optimally determine what to fill in each container in addition to the recipient selection decision.

In the following, we first discuss how the best bundle assumption can be adapted to capture reservations of some critical products for future recipients through exogenous quantity limits. Then, we present a heuristic approach to relax the best bundle assumption so that what the selected recipient will receive also depends on the needs of other recipients in the system. Finally, we discuss the challenges faced by the general problem from both implementation and computation perspectives.

First, in the case of MedShare, we observe that each recipient usually receives one or at most two pieces of each critical biomedical equipment (even if MedShare has a few more in inventory), so as to reserve the rest for future recipients. In this case, the best bundle of each recipient is defined as the set of items that maximizes the container value among the truncated MSRO inventory, where the inventory is truncated by quantity limits the MSROs imposes. This is how we constructed the calibrated numerical study in §7 and how the best bundle assumption is implemented in practice.

Next, we present a simple and intuitive heuristic that further relaxes the best bundle assumption. In particular, we stick with our proposed recipient selection approach; once a recipient i is selected, we determine the container mix for recipient i as follows: Let $\gamma_{-i,j}$ the the average ranking of all remaining recipients waiting in the system for product j; then, instead of sending the best bundle to recipient i, we skip product j (i.e., reserve it for future recipients) if $\mu_j = 1$ and $\gamma_{i,j} - \gamma_{-i,j} \geq \Delta\gamma$, where $\Delta\gamma$ is a policy parameter. Clearly, when $\Delta\gamma \geq N$, this approach reduces to the best bundle assumption. Otherwise, what this approach does is to reserve the products that are significantly higher ranked by other recipients (i.e., when the gap between the rankings is greater than or equal to $\Delta\gamma$).

Under the same setup we considered in §7, we numerically test the performance of the above approach under different $\Delta\gamma$ values and report the total value provision to recipients below. From these results, we observe that the proposed heuristic to relax the best bundle assumption has some value added over the best bundle approach (a performance improvement from 6.7% to 7.5% when $\Delta\gamma = 1$ and 2).

$\Delta\gamma$	0	1	2	3	$N = 20$
Improvement over random priority	7.1%	7.5%	7.5%	7.3%	6.7%

While we stick to the proposed recipient selection mechanism and relax the best bundle assumption for the selected recipient in the heuristic presented above, we note the most general problem is to simultaneous optimize which recipient to serve and what products to send to the selected recipient. However, the general problem is challenging from both implementation and analytical tractability perspectives. First, if what each recipient receives is determined by the ranking profile of all recipients, it becomes unclear how a scoring approach for recipient selection will continue to work. More specifically, if one is interested in sticking to a scoring approach, then intuitively, the score of each recipient should depend on not only her own ranking but also the rankings of other recipients, which makes this approach more complicated and less intuitive (note that in other settings such as organ allocation, the score of each patient does not depend on the characteristics of other patients). Second, from a pure computation perspective, the general problem is challenging as even the centralized version of the problem (i.e., assume recipient valuations are known to the MSRO) is intractable due to the high dimensionality of both state and action spaces. Hence, finding competitive yet simple and implementable solutions that simultaneously determine which recipient to serve at each shipping opportunity and which products to allocate to the chosen recipient is an interesting problem that we leave for future research.

C.3 Description of Date Set and Calibrated Numerical Study

The data set from MedShare contains the information about 39 containers shipped between July 2015 and April 2016. During this period, MedShare shipped out a total of more than one hundred different kinds of biomedical equipment. For each shipped product, the data set contains detailed information about recipient name, transportation mode, container capacity, and donor value (i.e., market value of the allocated product), among others. The data set also contains two timestamps that are critical for us to estimate the inventory levels and arrival rates of each product: "Created" and "ConfirmedDate". The "Created" timestamp represents the time when the product became available in MedShare's inventory, while the "ConfirmedDate" timestamp represents time when the product was secured for shipment to a specific recipient (for products that are still in MedShare's inventory, the "ConfirmedDate" timestamp is empty). Below, we describe how we construct the calibrated numerical study based on this data set.

Inventory Levels. We divide the interval from July 2015 to April 2016 into 39 periods based on the shipment date of the 39 containers. Since we have the "Created" and "ConfirmedDate" timestamps for all products that are shipped from July 2015 to April 2016 and the "Created" timestamps for all products that were still in MedShare's inventory at the end of April 2016, we are able to compute inventory levels at each period from July 2015 to April 2016. In particular, we first determine the initial inventory levels at the beginning of July 2015 using the "Created" timestamps for all products that had arrived and were not shipped out before July 2015. Then, we determine the inventory levels at each period by incorporating the inflow through the "Created" timestamps, and outflow through the "ConfirmedDate" timestamps.

Arrival Rates. To determine the arrival distribution of each product, we first determine the arrival quantity of each product during each period using the "Created" timestamps. Based on this arrival data, we assume that the arrival of each product follows a Poisson process, and estimate the arrival rate of each product by calculating the mean of the arrival quantities during all 39 periods.

Valuations. Finally, we describe how we construct recipients' valuations. A natural measure for the valuation of each product is the donor value, i.e., the donor's estimate of the market value of a donated product. However, an examination of the shipment data reveals that recipients do not necessarily pick the highest market value products, which suggests that recipient needs may sometimes favor ordering lower market value products. Therefore, we next describe a recipient-choice based adjustment to donor values to account for recipient-specific, needs-based valuations of different product categories.

First, we analyze the inventory level of each product before each recipient's order. If a product is available in inventory (i.e., it has a positive inventory level) and is picked by a recipient, we say that this product is critical to this recipient. If a product is available in inventory but is not picked by this recipient, we say that this product is noncritical to this recipient. For products that are not available (i.e., have a zero inventory level) before a recipient orders, we assume that there is a probability q for each of these products to be critical, and we estimate q as the fraction of critical products among the products with positive inventory levels, which is estimated to be around 0.4 in our numerical study.[1] Finally, after all products are designated as either critical or noncritical for a given recipient, we set the recipient-specific valuations in the following way such that the valuation of a noncritical product is no higher than that of a critical product. In particular, we first set the valuation of a critical product for this recipient as the donor value of this product. For a noncritical product, if its donor value is lower than or equal to the smallest donor value of all critical products, we set the valuation as its donor value; otherwise, we set the valuation as the smallest donor value of critical products. An illustrative example is as follows:

Suppose MedShare has a total of five product categories in the system, and the donor values for products 1-5 are 30, 25, 40, 20, and 35, respectively. Suppose the inventory levels are positive for all products when recipient A orders, and recipient A has picked products 1-3 but did not pick products 4 and 5. Then, we say products 1-3 are critical while products 4 and 5 are noncritical to recipient A. Therefore, we simply set recipient A's valuations for products 1-3 as their donor values. For product 4, since its donor value is $20 < \min(30, 25, 40) = 25$, we set recipient A's valuation

[1]Through extensive numerical analyses, we find that the value improvement of the proposed mechanisms over random priority decreases in q. As we reported in the main text, when q is estimated as the fraction of critical products among the products with positive inventory levels, mechanism $NI\text{-}NC$ improves the total value provision by 6.7%. In the worst-case when $q = 1$, our numerical analysis indicates that this value provision improvement would be 4.8%.

for product 4 as its donor value 20. Finally, for product 5, since its donor value is $35 > 25$, we set the valuation for product 5 as 25. Hence, recipient A's valuations for products 1-5 are set as 30, 25, 40, 20, and 25, respectively.

The above procedure provides a set of recipient-specific valuations that are closely related to the donor values and are consistent with recipients' past picking behavior (if a recipient values a noncritical product more than a critical product, she would have picked a noncritical product instead of a critical one, which contradicts the definition of critical and noncritical products). While there could be multiple ways of constructing recipient valuations and our proposed approach is not unique, we note that it provides a reasonable set of recipient-specific valuations that are consistent with recipients' picking behaviors and that it allows for a systematic comparison of the value provision under different mechanisms.

REFERENCES

[1] US Department of Health and Human Services, *The 2011 national blood collection and utilization survey report*, 2013.

[2] Grocery Manufacturer Association, *2008 joint industry unsaleables report: The real causes and actionable solutions*, 2008.

[3] P. Gregor, R. Forthofer, and A. Kapadia, "An evaluation of inventory and transportation policies of a regional blood distribution system," *European Journal of Operational Research*, vol. 10, no. 1, pp. 106–113, 1982.

[4] R. Kopach, D. Frances, and S. Sadat, "Models for predicting critical blood product shortages," in *Proceedings of the 29th Meeting of the European Working Group on Operational Research Applied to Health Services, ORAHS*, 2003, pp. 77–90.

[5] K. Katsaliaki, "Cost-effective practices in the blood service sector," *Health policy*, vol. 86, no. 2, pp. 276–287, 2008.

[6] B. Fries, "Optimal ordering policy for a perishable commodity with fixed lifetime," Cornell University, Ithaca, NY, Tech. Rep., 1972.

[7] W. Pierskalla and C. Roach, "Optimal issuing policies for perishable inventory," *Management Science*, vol. 18, no. 11, pp. 603–614, 1972.

[8] S. Nahmias, "Perishable inventory theory: A review," *Operations Research*, vol. 30, no. 4, pp. 680–708, 1982.

[9] ——, *Perishable inventory systems*. International Series in Operations Research and Management Science, Vol. 160. Springer, 2011.

[10] I. Karaesmen, A. Scheller-Wolf, and B. Deniz, "Managing perishable and aging inventories: Review and future research directions," in *Planning Production and Inventories in the Extended Enterprise*, Springer, 2011, pp. 393–436.

[11] B. Fries, "Optimal ordering policy for a perishable commodity with fixed lifetime," *Operations Research*, vol. 23, no. 1, pp. 46–61, 1975.

[12] S. Nahmias, "Optimal ordering policies for perishable inventory-II," *Operations Research*, vol. 23, no. 4, pp. 735–749, 1975.

[13] ——, "Myopic approximations for the perishable inventory problem," *Management Science*, vol. 22, no. 9, pp. 1002–1008, 1976.

[14] M. Cohen, "Analysis of single critical number ordering policies for perishable inventories," *Operations Research*, vol. 24, no. 4, pp. 726–741, 1976.

[15] D. Chazan and S. Gal, "A markovian model for a perishable product inventory," *Management Science*, vol. 23, no. 5, pp. 512–521, 1977.

[16] P. Nandakumar and T. Morton, "Near myopic heuristics for the fixed-life perishability problem," *Management Science*, vol. 39, no. 12, pp. 1490–1498, 1993.

[17] W. Cooper, "Pathwise properties and performance bounds for a perishable inventory system," *Operations Research*, vol. 49, no. 3, pp. 455–466, 2001.

[18] Y. Li, A. Lim, and B. Rodrigues, "Note-pricing and inventory control for a perishable product," *Manufacturing & Service Operations Management*, vol. 11, no. 3, pp. 538–542, 2009.

[19] X. Chen, Z. Pang, and L. Pan, "Coordinating inventory control and pricing strategies for perishable products," *Operations Research*, vol. 62, no. 2, pp. 284–300, 2014.

[20] H. Zhang, C. Shi, and X. Chao, "Nonparametric learning algorithms for optimal base-stock policy in perishable inventory systems with censored demand," in *Working Paper*, 2016.

[21] R. Broekmeulen and K. van Donselaar, "A heuristic to manage perishable inventory with batch ordering, positive lead-times, and time-varying demand," *Computers & Operations Research*, vol. 36, no. 11, pp. 3013–3018, 2009.

[22] E. Brodheim, C. Derman, and G. Prastacos, "On the evaluation of a class of inventory policies for perishable products such as blood," *Management Science*, vol. 21, no. 11, pp. 1320–1325, 1975.

[23] B. Deniz, I. Karaesmen, and A. Scheller-Wolf, "Managing perishables with substitution: Inventory issuance and replenishment heuristics," *Manufacturing & Service Operations Management*, vol. 12, no. 2, pp. 319–329, 2010.

[24] S. Nahmias, "Higher-order approximations for the perishable-inventory problem," *Operations Research*, vol. 25, no. 4, pp. 630–640, 1977.

[25] R. Levi, M. Pál, R. Roundy, and D. Shmoys, "Approximation algorithms for stochastic inventory control models," *Mathematics of Operations Research*, vol. 32, no. 2, pp. 284–302, 2007.

[26] R. Levi, G. Janakiraman, and M. Nagarajan, “A 2-approximation algorithm for stochastic inventory control models with lost sales,” *Mathematics of Operations Research*, vol. 33, no. 2, pp. 351–374, 2008.

[27] R. Levi, R. Roundy, D. Shmoys, and V. Truong, “Approximation algorithms for capacitated stochastic inventory control models,” *Operations Research*, vol. 56, no. 5, pp. 1184–1199, 2008.

[28] R. Levi and C. Shi, “Approximation algorithms for the stochastic lot-sizing problem with order lead times,” *Operations Research*, vol. 61, no. 3, pp. 593–602, 2013.

[29] C. Shi, H. Zhang, X. Chao, and R. Levi, “Approximation algorithms for capacitated stochastic inventory systems with setup costs,” *Naval Research Logistics*, vol. 61, no. 4, pp. 304–319, 2014.

[30] Z. Tao and S. Zhou, “Approximation balancing policies for inventory systems with remanufacturing,” *Mathematics of Operations Research*, vol. 39, no. 4, pp. 1179–1197, 2014.

[31] X. Chao, X. Gong, C. Shi, C. Yang, H Zhang, and S. Zhou, “Approximation algorithms for capacitated perishable inventory systems with positive lead times,” in *Working Paper*, 2016.

[32] X. Chao, X. Gong, C. Shi, and H. Zhang, “Approximation algorithms for perishable inventory systems,” *Operations Research*, vol. 63, no. 3, pp. 585–601, 2015.

[33] H. Zhang, C. Shi, and X. Chao, “Approximation algorithms for perishable inventory systems with setup costs,” *Operations Research*, forthcoming, 2016.

[34] A. Muharremoglu and J. Tsitsiklis, “A single-unit decomposition approach to multiechelon inventory systems,” *Operations Research*, vol. 56, no. 5, pp. 1089–1103, 2008.

[35] R. Haijema, J. van der Wal, and N. van Dijk, “Blood platelet production: Optimization by dynamic programming and simulation,” *Computers & Operations Research*, vol. 34, no. 3, pp. 760–779, 2007.

[36] D. Zhou, L. Leung, and W. Pierskalla, “Inventory management of platelets in hospitals: Optimal inventory policy for perishable products with regular and optional expedited replenishments,” *Manufacturing & Service Operations Management*, vol. 13, no. 4, pp. 420–438, 2011.

[37] L. Alwan, M. Xu, D. Yao, and X Yue, “The dynamic newsvendor model with correlated demand,” *Decision Sciences*, vol. 47, no. 1, pp. 11–30, 2016.

[38] N. Agrawal and S. Smith, “Multi-location inventory models for retail supply chain management,” in *Retail Supply Chain Management*, Springer, 2015, pp. 319–347.

[39] S. Allen, “Redistribution of total stock over several user locations,” *Naval Research Logistics (NRL)*, vol. 5, no. 4, pp. 337–345, 1958.

[40] J. Yang and Z. Qin, “Capacitated production control with virtual lateral transshipments,” *Operations Research*, vol. 55, no. 6, pp. 1104–1119, 2007.

[41] X. Hu, I. Duenyas, and R. Kapuscinski, “Optimal joint inventory and transshipment control under uncertain capacity,” *Operations Research*, vol. 56, no. 4, pp. 881–897, 2008.

[42] H. Abouee-Mehrizi, O. Berman, and S. Sharma, “Optimal joint replenishment and transshipment policies in a multi-period inventory system with lost sales,” *Operations Research*, vol. 63, no. 2, pp. 342–350, 2015.

[43] X. Chen, X. Gao, and Z. Hu, “A new approach to two-location joint inventory and transshipment control via l-convexity,” *Operations Research Letters*, vol. 43, no. 1, pp. 65–68, 2015.

[44] T. Cheong, “Joint inventory and transshipment control for perishable products of a two-period lifetime,” *The International Journal of Advanced Manufacturing Technology*, vol. 66, no. 9-12, pp. 1327–1341, 2013.

[45] B. Deniz, A. Scheller-Wolf, and I. Karaesmen, “Managing inventories of perishable goods: The effect of substitution,” in, Working paper, Carnegie-Mellon University, Pittsburgh, PA, 2004.

[46] H. Zhang, C. Shi, and X. Chao, “Nonparametric learning algorithms for optimal base-stock policy in perishable inventory systems with censored demand,” in, Working paper, University of Michigan, Ann Arbor, MI, 2017.

[47] L. Robinson, “Optimal and approximate policies in multiperiod, multilocation inventory models with transshipments,” *Operations Research*, vol. 38, no. 2, pp. 278–295, 1990.

[48] G. Tagaras and M. Cohen, “Pooling in two-location inventory systems with non-negligible replenishment lead times,” *Management Science*, vol. 38, no. 8, pp. 1067–1083, 1992.

[49] C. Paterson, G. Kiesmüller, R. Teunter, and K. Glazebrook, “Inventory models with lateral transshipments: A review,” *European Journal of Operational Research*, vol. 210, no. 2, pp. 125–136, 2011.

[50] X. Huang, "A review on policies and supply chain relationships under inventory transshipment," *Bulletin of Statistics and Operational Research*, vol. 29, no. 1, pp. 21–42, 2013.

[51] D. Gross, "Centralized inventory control in multilocation supply systems," *Multi-stage Inventory Models and Techniques*, vol. 1, p. 47, 1963.

[52] C. Das, "Supply and redistribution rules for two-location inventory systems: One-period analysis," *Management Science*, vol. 21, no. 7, pp. 765–776, 1975.

[53] U. Karmarkar, "The multiperiod multilocation inventory problem," *Operations Research*, vol. 29, no. 2, pp. 215–228, 1981.

[54] ——, "The multilocation multiperiod inventory problem: Bounds and approximations," *Management Science*, vol. 33, no. 1, pp. 86–94, 1987.

[55] G. Tagaras, "Effects of pooling on the optimization and service levels of two-location inventory systems," *IIE transactions*, vol. 21, no. 3, pp. 250–257, 1989.

[56] R. Lien, S. Iravani, K. Smilowitz, and M. Tzur, "An efficient and robust design for transshipment networks," *Production and Operations Management*, vol. 20, no. 5, pp. 699–713, 2011.

[57] C. Liang, S. Sethi, R. Shi, and J. Zhang, "Inventory sharing with transshipment: Impacts of demand distribution shapes and setup costs," *Production and Operations Management*, vol. 23, no. 10, pp. 1779–1794, 2014.

[58] D. Yao, S. Zhou, and W. Zhuang, "Joint initial stocking and transshipmentasymptotics and bounds," *Production and Operations Management*, vol. 25, no. 2, pp. 273–289, 2016.

[59] N. Rudi, S. Kapur, and D. Pyke, "A two-location inventory model with transshipment and local decision making," *Management science*, vol. 47, no. 12, pp. 1668–1680, 2001.

[60] H. Zhao, V. Deshpande, and J. Ryan, "Inventory sharing and rationing in decentralized dealer networks," *Management Science*, vol. 51, no. 4, pp. 531–547, 2005.

[61] X. Huang and G. Sošić, "Transshipment of inventories: Dual allocations vs. transshipment prices," *Manufacturing & Service Operations Management*, vol. 12, no. 2, pp. 299–318, 2010.

[62] P. Sun, K. Wang, and P. Zipkin, "Quadratic approximation of cost functions in lost sales and perishable inventory control problems," in, Fuqua School of Business, Duke University, Durham, NC, 2014.

[63] H. Zhang, C. Shi, and X. Chao, "Technical note-approximation algorithms for perishable inventory systems with setup costs," *Operations Research*, vol. 64, no. 2, pp. 432–440, 2016.

[64] X. Chao, X. Gong, C. Shi, C. Yang, H Zhang, and S. Zhou, "Approximation algorithms for capacitated perishable inventory systems with positive lead times," Working paper, University of Michigan, Ann Arbor, MI, 2017.

[65] C. Zhang, T. Ayer, and C. White, "2-approximation algorithms for perishable inventory control when FIFO is an optimal issuing policy.," Working paper, Georgia Institute of Technology, Atlanta, GA, 2017.

[66] M. Fontaine, Y. Chung, W. Rogers, H. Sussmann, P. Quach, S. Galel, L. Goodnough, and F. Erhun, "Improving platelet supply chains through collaborations between blood centers and transfusion services," *Transfusion*, vol. 49, no. 10, pp. 2040–2047, 2009.

[67] K. Wang and Z. Ma, "Age-based policy for blood transshipment during blood shortage," *Transportation Research Part E: Logistics and Transportation Review*, vol. 80, pp. 166–183, 2015.

[68] G. Prastacos, "Optimal myopic allocation of a product with fixed lifetime," *Journal of the Operational Research Society*, vol. 29, no. 9, pp. 905–913, 1978.

[69] G. Prastacos and E. Brodheim, "Pbds: A decision support system for regional blood management," *Management Science*, vol. 26, no. 5, pp. 451–463, 1980.

[70] G. Prastacos, "Allocation of a perishable product inventory," *Operations Research*, vol. 29, no. 1, pp. 95–107, 1981.

[71] M. Cohen, W. Pierskalla, and H. Yen, "An analysis of ordering and allocation policies for multi-echelon, age-differentiated inventory systems," *TIMS Studies in the Management Sciences*, vol. 16, no. 1, pp. 353–378, 1981.

[72] M. Cohen, W. Pierskalla, R. Sassetti, and J. Consolo, "An overview of a hierarchy of planning models for regional blood bank management," *Transfusion*, vol. 19, no. 5, pp. 526–534, 1979.

[73] J. Lang, "Blood bank inventory control with transshipments and substitutions," in *Production and Inventory Management with Substitutions*, Springer, 2010, pp. 205–226.

[74] K. Wang, Z. Ma, and Y. Zhou, "Age-based transshipment strategy for overstock blood in emergency relief," *Advances in Information Sciences & Service Sciences*, vol. 4, no. 18, 2012.

[75] M. Jin and G. Agirbas, “Reducing expiry of slow-moving critical medical items with transshipment,” *IIE Transactions on Healthcare Systems Engineering*, vol. 3, no. 3, pp. 193–206, 2013.

[76] Y. Lu and J. Song, “Order-based cost optimization in assemble-to-order systems,” *Operations Research*, vol. 53, no. 1, pp. 151–169, 2005.

[77] P. Zipkin, “On the structure of lost-sales inventory models,” *Operations Research*, vol. 56, no. 4, pp. 937–944, 2008.

[78] W. Huh and G. Janakiraman, “On the optimal policy structure in serial inventory systems with lost sales,” *Operations Research*, vol. 58, no. 2, pp. 486–491, 2010.

[79] Z. Pang, F. Chen, and Y. Feng, “Technical note-a note on the structure of joint inventory-pricing control with leadtimes,” *Operations Research*, vol. 60, no. 3, pp. 581–587, 2012.

[80] Q. Li and P. Yu, “Multimodularity and its applications in three stochastic dynamic inventory problems,” *Manufacturing & Service Operations Management*, vol. 16, no. 3, pp. 455–463, 2014.

[81] K. Murota, *Discrete convex analysis*. SIAM, 2003.

[82] W. Chen, M. Dawande, and G. Janakiraman, “Fixed-dimensional stochastic dynamic programs: An approximation scheme and an inventory application,” *Operations Research*, vol. 62, no. 1, pp. 81–103, 2014.

[83] A. Berry, “The power of one: MedShare 2014 annual report,” in, Medshare, Decatur, GA, 2014.

[84] World Health Organization, “Children: Reducing mortality.,” in, http://www.who.int /mediacentre/factsheets/fs178/en/, 2016.

[85] ——, “Medical devices: Managing the mismatch: An outcome of the priority medical devices project.,” in, WHO Press, Switzerland, 2010.

[86] A. Pedraza Martinez, S. Hasija, and L. Van Wassenhove, “An operational mechanism design for fleet management coordination in humanitarian operations,” in, Working Paper, INSEAD Business School, Fontainebleau, France, 2010.

[87] A. Abdulkadiroğlu and T. Sönmez, “School choice: A mechanism design approach,” *The American Economic Review*, vol. 93, no. 3, pp. 729–747, 2003.

[88] M. Akan, O. Alagoz, B. Ata, F. Erenay, and A. Said, “A broader view of designing the liver allocation system,” *Operations Research*, vol. 60, no. 4, pp. 757–770, 2012.

[89] D. Bertsimas, V. Farias, and N. Trichakis, “Fairness, efficiency, and flexibility in organ allocation for kidney transplantation,” *Operations Research*, vol. 61, no. 1, pp. 73–87, 2013.

[90] X. Su and S. Zenios, “Recipient choice can address the efficiency-equity trade-off in kidney transplantation: A mechanism design model,” *Management Science*, vol. 52, no. 11, pp. 1647–1660, 2006.

[91] S. Gavirneni, R. Kapuscinski, and S. Tayur, “Value of information in capacitated supply chains,” *Management Science*, vol. 45, no. 1, pp. 16–24, 1999.

[92] G. Cachon and M. Fisher, “Supply chain inventory management and the value of shared information,” *Management Science*, vol. 46, no. 8, pp. 1032–1048, 2000.

[93] H. Lee, K. So, and C. Tang, “The value of information sharing in a two-level supply chain,” *Management Science*, vol. 46, no. 5, pp. 626–643, 2000.

[94] A. Abdulkadiroğlu, Y. Che, and Y. Yasuda, “Resolving conflicting preferences in school choice: The “Boston mechanism” reconsidered,” *The American Economic Review*, vol. 101, no. 1, pp. 399–410, 2011.

[95] M. Pycia, “The cost of ordinality,” in, Working paper, University of California, Los Angeles, CA, 2014.

[96] I. Ashlagi and P. Shi, “Optimal allocation without money: An engineering approach,” *Management Science*, vol. 62, no. 4, pp. 1078–1097, 2015.

[97] B. Compton, “Let’s make donations more effective: Don’t waste these important efforts.,” *Health Progress (Saint Louis, Mo.)*, vol. 93, no. 2, p. 84, 2012.

[98] P. Howitt, A. Darzi, G. Yang, H. Ashrafian, R. Atun, J. Barlow, A. Blakemore, A. Bull, J. Car, L. Conteh, *et al.*, “Technologies for global health,” *The Lancet*, vol. 380, no. 9840, pp. 507–535, 2012.

[99] M. Denizel, M. Ferguson, and L. Toktay, “Building sustainability into humanitarian aid operations: An efficient frontier approach to the selection of recipients for medical supplies donation,” *Operations Management Education Review*, vol. 7, pp. 49–78, 2013.

[100] T. Kotsi, L. Van Wassenhove, and M. Hensen, "Medicine donations: Matching demand with supply in broken supply chains," in, Working Paper, INSEAD Business School, Fontainebleau, France, 2014.

[101] J. MacRae, "The remedy utilization study: A model for donations evaluations," PhD thesis, Yale University, New Haven, CT, 1997.

[102] J. Dyro, "Donation of medical device technologies," *Clinical Engineering Handbook*, pp. 155–158, 2004.

[103] A. Atasu, B. Toktay, W. Yeo, and C. Zhang, "Effective medical surplus recovery," in, Forthcoming in *Production and Operations Management*, 2016.

[104] L. Van Wassenhove and A. Pedraza Martinez, "Using or to adapt supply chain management best practices to humanitarian logistics," *International Transactions in Operational Research*, vol. 19, no. 1-2, pp. 307–322, 2012.

[105] S. Gupta, M. Starr, R. Farahani, and N. Matinrad, "Disaster management from a POM perspective: Mapping a new domain," *Production and Operations Management*, vol. 25, no. 10, pp. 1611–1637, 2016.

[106] R. Lien, S. Iravani, and K. Smilowitz, "Sequential resource allocation for nonprofit operations," *Operations Research*, vol. 62, no. 2, pp. 301–317, 2014.

[107] K. Natarajan and J. Swaminathan, "Inventory management in humanitarian operations: Impact of amount, schedule, and uncertainty in funding," *Manufacturing & Service Operations Management*, vol. 16, no. 4, pp. 595–603, 2014.

[108] ——, "Multi-treatment inventory allocation in humanitarian health settings under funding constraints," in, Forthcoming in *Production and Operations Management*, 2016.

[109] J. Gallien, I. Rashkova, R. Atun, and P. Yadav, "National drug stockout risks and the global fund disbursement process for procurement," in, Forthcoming in *Production and Operations Management*, 2016.

[110] M. Dawande, S. Gavirneni, M. Mehrotra, and V. Mookerjee, "Efficient distribution of water between head-reach and tail-end farms in developing countries," *Manufacturing & Service Operations Management*, vol. 15, no. 2, pp. 221–238, 2013.

[111] M. Besiou, A. J Pedraza-Martinez, and L. Van Wassenhove, "Vehicle supply chains in humanitarian operations: Decentralization, operational mix, and earmarked funding," *Production and Operations Management*, vol. 23, no. 11, pp. 1950–1965, 2014.

[112] S. Bhattacharya, S. Hasija, and L. Van Wassenhove, "Designing efficient infrastructural investment and asset transfer mechanisms in humanitarian supply chains," *Production and Operations Management*, vol. 23, no. 9, pp. 1511–1521, 2014.

[113] Ö. Ergun, L. Gui, J. Heier Stamm, P. Keskinocak, and J. Swann, "Improving humanitarian operations through technology-enabled collaboration," *Production and Operations Management*, vol. 23, no. 6, pp. 1002–1014, 2014.

[114] T. Taylor and W. Xiao, "Subsidizing the distribution channel: Donor funding to improve the availability of malaria drugs," *Management Science*, vol. 60, no. 10, pp. 2461–2477, 2014.

[115] S. Zenios, G. Chertow, and L. Wein, "Dynamic allocation of kidneys to candidates on the transplant waiting list," *Operations Research*, vol. 48, no. 4, pp. 549–569, 2000.

[116] X. Su and S. Zenios, "Patient choice in kidney allocation: The role of the queueing discipline," *Manufacturing & Service Operations Management*, vol. 6, no. 4, pp. 280–301, 2004.

[117] X. Su and S. Zenios, "Patient choice in kidney allocation: A sequential stochastic assignment model," *Operations Research*, vol. 53, no. 3, pp. 443–455, 2005.

[118] J. Leshno, "Dynamic matching in overloaded waiting lists," in, Columbia University, New York, NY, 2015.

[119] N. Thakral, "The public-housing allocation problem: Theory and evidence from Pittsburgh," in, Working paper, Harvard University, Cambridge, MA, 2016.

[120] C. Prendergast, "The allocation of food to food banks," in, Working paper, University of Chicago Booth School of Business, Chicago, IL, 2017.

[121] G. Cachon and M. Lariviere, "Capacity allocation using past sales: When to turn-and-earn," *Management Science*, vol. 45, no. 5, pp. 685–703, 1999.

[122] ——, "Capacity choice and allocation: Strategic behavior and supply chain performance," *Management Science*, vol. 45, no. 8, pp. 1091–1108, 1999.

[123] L. Lu and M. Lariviere, "Capacity allocation over a long horizon: The return on turn-and-earn," *Manufacturing & Service Operations Management*, vol. 14, no. 1, pp. 24–41, 2012.

[124] E. Spiliotopoulou, K. Donohue, and M. Gürbüz, "Information reliability in supply chains: The case of multiple retailers," *Production and Operations Management*, vol. 25, no. 3, pp. 548–567, 2015.

[125] D. Bergemann and J. Valimaki, “Information in mechanism design,” in, Yale University, New Haven, CT, 2006.

[126] P. Milgrom and R. Weber, “A theory of auctions and competitive bidding,” *Econometrica: Journal of the Econometric Society*, pp. 1089–1122, 1982.

[127] A. Hylland and R. Zeckhauser, “The efficient allocation of individuals to positions,” *The Journal of Political Economy*, pp. 293–314, 1979.

[128] A. Miralles, “Cardinal bayesian allocation mechanisms without transfers,” *Journal of Economic Theory*, vol. 147, no. 1, pp. 179–206, 2012.

[129] A. Bogomolnaia and H. Moulin, “A new solution to the random assignment problem,” *Journal of Economic Theory*, vol. 100, no. 2, pp. 295–328, 2001.

[130] F. Kojima, “Random assignment of multiple indivisible objects,” *Mathematical Social Sciences*, vol. 57, no. 1, pp. 134–142, 2009.

[131] R. Myerson, “Incentive compatibility and the bargaining problem,” *Econometrica: Journal of the Econometric Society*, pp. 61–73, 1979.

[132] L. Schulman and V. Vazirani, “Allocation of divisible goods under lexicographic preferences,” in, ArXiv Preprint: https://arxiv.org/abs/1206.4366, 2015.

[133] D. Blair and R. Pollak, “Acyclic collective choice rules,” *Econometrica: Journal of the Econometric Society*, pp. 931–943, 1982.

[134] D. Black, “Partial justification of the borda count,” *Public Choice*, vol. 28, no. 1, pp. 1–15, 1976.

[135] A. Martin, M. Bartels, J. Hauss, and J. Fangmann, “Overview of the meld score and the unos adult liver allocation system,” *Transplantation Proceedings*, vol. 39, no. 10, pp. 3169–3174, 2007.

[136] G. Hurley, P. Jackson, R. Levi, R. Roundy, and D. Shmoys, “New policies for stochastic inventory control models–theoretical and computational results,” Cornell University, Ithaca, NY, Tech. Rep., 2007.